# Holographic Entanglement Entropy

# Introduction

•AdS/CFT Correspondence→statement of duality between $\mathcal{N} = 4$ Super Yang Mills in 4 dimensions and Type IIB Superstring Theory On $AdS_5 x S^5$ clearly showcases the Holographic Principle.

•One can use this holographic aspect of AdS/CFT correspondence to investigate strongly coupled Conformal Field Theory by computing quantum effects using Classical Gravity prescription.

•If one wishes to understand the properties of a given QFT then understanding of non local quantities cannot be neglected.

•One such example of a non local quantity is Entanglement Entropy, however calculation of this quantity is not easy in certain cases and it is here that we can use Gravitational tools in the background of Holography to calculate the so called Holographic Entanglement Entropy.

# Motivation for AdS/CFT Correspondence

1. Large N expansion
2. Holographic Principle
3. D Branes
4. Scattering Cross-section

# Large N expansion

•In the low energy limit if one wishes to study Strong interactions, QCD becomes strongly coupled and perturbative analysis becomes difficult.

•It would be helpful at this stage if there would exist a weakly coupled theory dual to QCD to have a better understanding of low energy physics of Strong Interactions.

•It was 't Hooft who first pointed out that there exists a duality between String Theory and QCD via Large N expansion.

•'t Hooft studied Gauge Theory based on Gauge group SU(N) in the limit N $\to \infty$ with 1/N as the expansion parameter. Effective coupling constant in this case would be $\lambda_{th} = g_{YM}^2 N$ in which $\lambda_{th}$ is held fixed and is called 't Hooft limit where $g_{YM}^2 \to 0$.

•As a result, we get a factor of N/$\lambda_{th}$ infront of the entire Lagrangian

$$\mathcal{L} = \frac{N}{\lambda_{th}}[TrF^2_{\mu\nu} + ...]$$

•Analysis of the Feynman diagrams which are represented using Double line notation led to the fact that the form of the perturbative expansion is similar to that of closed strings and one can indentify 1/N as the string coupling constant.

# Holographic principle

•Black Hole Entropy Problem $\rightarrow$ When some matter is thrown inside the black hole then since the observer outside has no clue as to what is happening inside the black hole,from the observer's perspective violation of second law of thermodynamics takes place.
•To solve the entropy problem related to black holes Bekenstein proposed that even black holes have entropy to preserve the second law of thermodynamics.This development led to Generalised Second Law of Thermodynamics which says that the combined entropy of the black hole and material outside the black hole never decreases.
•Hawking and Bekenstein independently were able to infer that black hole entropy is proportional to the area of the horizon and this result gave way to the Holographic Principle.
•Holographic Principle - A Quantum Gravity Theory in D dimensional spacetime can be described by a theory on it's D-1 dimensional boundary having less than one degree of freedom per planck area.

# D Branes

•Open Strings can have two kinds of Boundary conditions-Neumann(free endpoint) and Dirichlet(fixed endpoint) and with Dirichlet boundary condition we say that the endpoint lies on a hyperplane called as D brane.

•Presence of massless spectrum of open strings $\rightarrow$ U(1) Gauge Theory on a single D brane.

•We now consider N coincident parallel D branes and since open strings can start and end on different branes, there are $N^2$ species of open strings $\rightarrow$ $N^2$ is the dimension of the adjoint representation of U(N). So we have U(N) Gauge theory in this case.

•Now if these Branes are on top of each other ,implying no seperation between them then this stack becomes a heavy object which can curve space.

•This macroscopic object can be described in terms of Ramond-Ramond p+1 form potential and this entire stack is better known as Black Brane or stated in other words they are black hole type solutions in string theory.

•So we have two descriptions of a stack D branes

1. In terms of U(N) Gauge Theory

2. As classical solutions to Supergravity

# Scattering Cross section

•People studied scattering of particles from D Branes in the low energy limit and they found that the closed strings scattered freely implying that the scattering cross section was very low.

•These calculations were done at weak string coupling where the Gauge theory description due to open strings become valid.

•Next, people also studied scattering of particles from Black Brane and found that they also scatter off freely from the near horizon limit.

•These results were reflective of the underlying fact that considering only tree level calculation from Field theory side both the descriptions gave the same absorption cross section.

# AdS/CFT Correspondence

•We consider N parallel D3 branes on which open strings end in 10 dimensional spacetime. There will be two types perturbative excitations in this case:-

1. Closed String excitations in the bulk $\rightarrow$ in the low energy limit massless states are excited $\rightarrow$ effective Lagrangian is that of type IIB Supergravity.

2. Open String excitations on the D brane $\rightarrow$ again consider low energy limit where massless states are excited $\rightarrow$ effective Lagrangian is that of $\mathcal{N} = 4$ U(N) Super Yang Mills Theory.

•Along with this there there will be some interaction between the bulk and the brane but in the low energy limit this interaction tends to zero.

•Hence, we are left with two decoupled systems:-

1. Free Supergravity in the bulk

2.$\mathcal{N} = 4$ U(N) Super Yang Mills on D3 Branes

•As mentioned before , a stack of N parallel D3 Branes also behaves like a black hole whose metric is given as

$\mathrm{ds}^2 = \mathrm{H}^{-1/2}(\mathrm{r})\eta_{\mu\nu}\mathrm{dx}^\mu\mathrm{dx}^\nu + \mathrm{H}^{-1/2}(\mathrm{r})\eta_{mn}\mathrm{dx}^m\mathrm{dx}^n$ where
$\mathrm{H} = 1 + (\mathrm{L}/\mathrm{r})^4$ and $\mathrm{L}^4 = 4\pi g_s N\alpha^2$

•In the small r limit, that is near the horizon the metric is

$$\mathrm{ds}^2 = (\mathrm{r}/\mathrm{L})^2\eta_{\mu\nu}\mathrm{dx}^\mu\mathrm{dx}^\nu + (\mathrm{L}/\mathrm{r})^2\mathrm{dr}^2 + \mathrm{L}^2\mathrm{d}\Omega^2$$

•First two terms are that of $\text{Ads}_5$ metric while the last term is that of $S^5$.So, one can conclude that the near horizon geometry is that of $AdS_5xS^5$.

•Here also we have two types of excitations.One that corresponds to the Bulk and other near the horizon but since the $\sigma_{absorb}$ goes as $\omega^3$,in the low energy limit these two excitations decouple from each other and we left with

1. Free Supergravity in the bulk
2. Low energy superstring theory on $AdS_5xS^5$.

•Conjecture $\rightarrow$ Comparing these two forms of decoupled systems and noticing that there are free closed strings propagating in the bulk in both cases(or Supergravity);we are led to the conclusion that the $\mathcal{N} = 4$ U(N)SYM field theory is dual to type IIB superstring on $AdS_5xS^5$.

# AdS/CFT Dictionary

So , here we will talk about Mapping of Certain Quantities from Field Theory to Bulk :-

1. Symmetry
2. Spacetime Directions
3. Scale/Radius Duality
4. Operator/Field Correspondence
5. Wilson Loops on Boundary
6. Entanglement Entropy on Boundary

# Holographic Entanglement Entropy

•Consider a quantum mechanical system with many degrees of freedom , at zero temperature.Thus the Quantum mechanical system is described by the pure ground state and we assume no degeneracy of the ground state, then the density matrix is given as

$$\rho_{tot} = |\psi\rangle\langle\psi|$$

In the above case the Von Neumann entropy of the system which is

$$S_{tot} = -tr(\rho_{tot} log \rho_{tot}) = 0$$

Next, dividing the system into two subsystems A and B , we write the total Hilbert space as a direct product of two spaces

$$H_{tot} = H_A \otimes H_B$$

The obeserver who can access only the subsytem A will feel as if the total system is described by the reduced density matrix $\rho_A = tr_B \rho_{tot}$ where the trace is taken only over the Hilbert space $H_B$.

•Next, we define the entanglement entropy of the subsystem A as the von Neumann entropy of the reduced density matrix $\rho_A$ as

$$S_A = -tr_A(\rho_A log \rho_A)$$

•Entanglement Entropy in QFT's
We consider a QFT on a d+1 dimensional manifold RxN where R is the time direction and N is some d dimensional spacelike manifold.We define the subsystem by a d dimensional submanifold A $\subset$ N at a fixed time and we call it's complememt submanifold B. The boundary of A, denoted by $\partial$A divides the manifold N into two submanifolds A and B.Then we can define the entanglement entropy $S_A$ by

$$S_A = \gamma \frac{Area(\partial A)}{a^{d-1}} + \quad subleading \quad terms$$

where $\gamma$ is a constant which depends on the system.The above formula is called as the Area Law for the Entanglement Entropy.

•This law looks very similar to Bekenstein-Hawking entropy of black holes which is proportional to Area of the horizon.Rather,the motivation for Holographic Entanglement Entropy came from the entropy of a black hole which is proportional to the area of the horizon, which in turn gave way to the idea of Holography.Progress in the direction of idea of Holography was further enhanced by Ads/Cft correspondence which is a crucial part of Holographic Entanglement Entropy .

•To describe the holographic Entanglement entropy from Ads/Cft correspondence ,where a d+1 dimensional Cft lives on the boundary of a d+2 dimensional Ads space, what we do is divide the boundary denoted by N into two types of subsystems A and B and then to calculate entropy from the perspective of gravity we will use the poincare metric , setting $\mathrm{N} = \mathrm{R}^d$.

•Next, we extend this division(seperating the two systems) inside the bulk spacetime.

•This division or boundary of the subsystem A is denoted by $\partial A = \partial\gamma$, where $\gamma$ is the extension of the boundary inside the bulk and is a codimension 2 surface.Now the key point here is we can have many possible surfaces inside the bulk but we choose that surface which has the minimum area. The next step following this procedure is the formula for Holographic Entanglement Entropy which was proposed as

•

$$S_A = \frac{Area(\gamma_A)}{4G_N^{d+2}}$$

where ${G_N}^{d+2}$ is the d+2 dimensional Newton's constant in Ads Gravity.

# Sample Calculation for $Ads_3/Cft_2$ case

•We choose the poincare metric

$$ds^2 = (R/z)^2(dz^2 + dx^2 - dx_0^2)$$

•We define the subsystem A by the region -l/2$\leq$ x $\leq$ l/2 at the boundary z = 0 . Minimal surface here will be a geodesic.

•Now, the area is given as

$$dA = R\int_{-l/2}^{l/2} dx\sqrt{(1 + (dz/dx)^2)}/z$$

•Now, to make the $\sqrt{(1 + (dz/dx)^2)}$ independent of $x$ , we find out momentum canonical to z and then expecting the Hamiltonian to be independent of x we get the following condition:-

- $$\frac{dz}{dx} = \frac{\sqrt{c^2 - z^2}}{z}$$

- Integrating for x and z we get the following

$$x = l/2cos\theta, z = l/2sin\theta$$

- Now,plugging in the value for $\sqrt{1 + (dz/dx)^2}$ and dx , we get the following relation

$$dA = R \int \frac{l}{2z\sqrt{l^2/4 - z^2}} dz$$

•Now, subsituting the value for z and for dz/d$\theta$ we get with modified limits $\epsilon$ to $\pi/2$

$$dA = R \int_{\epsilon}^{\frac{\pi}{2}} \frac{d\theta}{sin\theta}$$

where $\epsilon \longrightarrow \frac{2a}{l}$

•On integrating this, we get

$$A = log[sin(\theta/2)] - log[cos(\theta/2)]$$

•Step wise now, put the limits , and then do a series expansion , finally plug the value of $\epsilon$ = 2a/l, to get A = $\frac{c}{3} log \frac{l}{a}$ where we have used the relation c = $\frac{3R}{2G_N^3}$.

•We get the same result when calculated from Cft side.

Thank you !