A new class of entropy-power-based uncertainty relations

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in collaboration with J.A. Dunningham  and A. Hayes 

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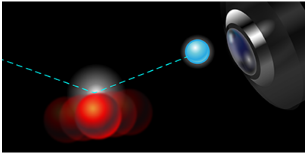
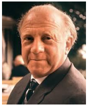
Outline



$$\langle \Delta p_i^2 \rangle_\psi \langle \Delta x_j^2 \rangle_\psi \ \geq \ \delta_{ij} \frac{\hbar^2}{4}$$

$$\mathcal{H}(\mathcal{P}^{(1)}) \ + \ \mathcal{H}(\mathcal{P}^{(2)}) \ \geq \ -2 \log c$$

Quantum-mechanical URs place fundamental limits on the accuracy with which one is able to measure values of different physical quantities. This has profound implications not only on the microscopic but also on the macroscopic level of physical description.



W. Heisenberg, *Physics and Beyond, 1971*

Outline

History I

1927 **Heisenberg's** intuitive derivation of UR $\delta p_x \, \delta x \approx \hbar$

1927 **Kennard** considers as δs as a standard deviation of s

1928 **Dirac** uses Hausdorff-Young's inequality to prove HUR. δx and δp_x are half-widths of wave packet and its Fourier image

1929/30 **Rebertson** and **Schrödinger** reinterpret HUR in terms of statistical ensemble of identically prepared experiments. Both δp and δx are standard deviations. Schwarz inequality in the proof.

1945 **Mandelstam** and **Tamm** derive time-energy UR

1947 **Landau** derives time-energy UR

1968 **Carruthers** and **Nietto** angle-angular momentum UR

History II

1969 **Hirschman** first Shannon's entr. based UR (weaker than VUR)

1971 **Synge's** three-observable UR

1976 **Lévy-Leblond** improves angle-angular momentum UR

1980 **Dodonov** derives mixed-states UR

$80-90'$**s** Most standard HUR's are re-derived from Cramér-Rao inequality using Fisher information

$1983/84$ **Deutsch** and **Białynicky-Birula** derive Shannon-entr.-based UR

$80-90'$**s** **Kraus, Maassen, etc.** derive Shannon-entropy-based UR with sharper bound than Deutsch and B-B

$00'$**s** **Uffink, Montgomery, Abe, etc.** derive other non-Shannonian UR

History III

2006/7s Ozawa's universal error-disturbance relations

2014 **Dressel–Nori** error-disturbance inequalities

2012 – 15 Violations of Heisenberg's UR measured by number of groups

nature physics
LETTERS
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Experimental demonstration of a universally valid error–disturbance uncertainty relation in spin measurements

Jacqueline Erhart[1], Stephan Sponar[1], Georg Sulyok[1], Gerald Badurek[1], Masanao Ozawa[2] and Yuji Hasegawa[1]*

The uncertainty principle generally prohibits simultaneous measurements of certain pairs of observables and forms the basis of indeterminacy in quantum mechanics[1]. Heisenberg's original formulation, illustrated by the famous γ-ray microscope, sets a lower bound for the product of the measurement error and the disturbance[2]. Later, the uncertainty relation was reformulated in terms of standard deviations[3–5], where the focus was exclusively on the indeterminacy of predictions, whereas the unavoidable recoil in measuring devices has been ignored[6]. A correct formulation of the error–disturbance uncertainty relation, taking recoil into account, is essential for a deeper understanding of the uncertainty principle, as Heisenberg's original relation is valid only under specific circumstances[7–10]. A new error–disturbance relation, derived using the theory of general quantum measurements, has been claimed to be universally valid[11–14]. Here, we report a neutron-optical experiment that records the error of a spin-component measurement as well as the disturbance caused on another spin-component. The results confirm that both error and disturbance obey the new relation but violate the old one in a wide range of an experimental parameter.

as $\sigma(A)^2 = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$. Note that a positive definite covariance term can be added to the right-hand side of equation (2), if squared, as discussed by Schrödinger[5]. For our experimental setting, this term vanishes. Robertson's relation (equation (2)) for standard deviations has been confirmed by many different experiments. In a single-slit diffraction experiment[15] the uncertainty relation, as expressed in equation (2), has been confirmed. A trade-off relation appears in squeezing coherent states of radiation fields[16], and many experimental demonstrations have been carried out[17].

Robertson's relation (equation (2)) has a mathematical basis, but has no immediate implications for limitations on measurements. This relation is naturally understood as limitations on state preparation or limitations on prediction from the past. On the other hand, the proof of the reciprocal relation for the error $\epsilon(A)$ of an A measurement and the disturbance $\eta(B)$ on observable B caused by the measurement, in a general form of Heisenberg's error–disturbance relation

$$\epsilon(A)\eta(B) \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \qquad (3)$$

is not straightforward, as Heisenberg's proof[2] used an unsupported

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PHYSICAL REVIEW A **88**, 022110 (2013)

Violation of Heisenberg's error-disturbance uncertainty relation in neutron-spin measurements

Georg Sulyok,[1] Stephan Sponar,[1] Jacqueline Erhart,[1] Gerald Badurek,[1] Masanao Ozawa,[2] and Yuji Hasegawa[1]
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(Received 3 June 2013; published 14 August 2013)

In its original formulation, Heisenberg's uncertainty principle dealt with the relationship between the error of a quantum measurement and the thereby induced disturbance on the measured object. Meanwhile, Heisenberg's heuristic arguments have turned out to be correct only for special cases. An alternative universally valid relation was derived by Ozawa in 2003. Here, we demonstrate that Ozawa's predictions hold for projective neutron-spin measurements. The experimental inaccessibility of error and disturbance claimed elsewhere has been overcome using a tomographic method. By a systematic variation of experimental parameters in the entire configuration space, the physical behavior of error and disturbance for projective spin-$\frac{1}{2}$ measurements is illustrated comprehensively. The violation of Heisenberg's original relation, as well as the validity of Ozawa's relation become manifest. In addition, our results conclude that the widespread assumption of a reciprocal relation between error and disturbance is not valid in general.

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I. INTRODUCTION

The uncertainty principle, proposed by Heisenberg [1] in 1927, ranks without doubt among the most famous statements

does not hold generally. Thus, his argument did not establish the universal validity of Eq. (1).

In 1929, Robertson [19] extended Kennard's relation, Eq. (2), to an arbitrary pair of observables A and B as

History III

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2012 – 15 Violations of Heisenberg's UR measured by number of groups

PRL **109**, 100404 (2012) PHYSICAL REVIEW LETTERS week ending 7 SEPTEMBER 2012

Violation of Heisenberg's Measurement-Disturbance Relationship by Weak Measurements

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(Received 4 July 2012; published 6 September 2012; publisher error corrected 23 October 2012)

While there is a rigorously proven relationship about uncertainties intrinsic to any quantum system, often referred to as "Heisenberg's uncertainty principle," Heisenberg originally formulated his ideas in terms of a relationship between the precision of a *measurement* and the disturbance it must create. Although this latter relationship is not rigorously proven, it is commonly believed (and taught) as an aspect of the broader uncertainty principle. Here, we experimentally observe a violation of Heisenberg's "measurement-disturbance relationship", using weak measurements to characterize a quantum system before and after it interacts with a measurement apparatus. Our experiment implements a 2010 proposal of Lund and Wiseman to confirm a revised measurement-disturbance relationship derived by Ozawa in 2003. Its results have broad implications for the foundations of quantum mechanics and for practical issues in quantum measurement.

DOI: 10.1103/PhysRevLett.109.100404 PACS numbers: 03.65.Ta, 03.67.Ac, 42.50.Xa

The Heisenberg uncertainty principle is one of the cornerstones of quantum mechanics. In his original paper on the subject, Heisenberg wrote, "At the instant of time when the position is determined, that is, at the instant when the photon is scattered by the electron, the electron undergoes

must satisfy $\epsilon(q)\eta(p) \approx h$, where h is Planck's constant. This idea was at the crux of the Bohr-Einstein debate [9], and the role of momentum disturbance in destroying interference has remained a subject of heated discussion [10–12]. Recently, the study of uncertainty relations in

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Experimental violation and reformulation of the Heisenberg's error-disturbance uncertainty relation

So-Young Baek[1]*, Fumihiro Kaneda[1], Masanao Ozawa[2] & Keiichi Edamatsu[1]

[1]Research Institute of Electrical Communication, Tohoku University, Sendai 980-8577, Japan, [2]Graduate School of Information Science, Nagoya University, Nagoya 464-8601, Japan.

The uncertainty principle formulated by Heisenberg in 1927 describes a trade-off between the error of a measurement of one observable and the disturbance caused on another complementary observable such that their product should be no less than the limit set by Planck's constant. However, Ozawa in 1988 showed a model of position measurement that breaks Heisenberg's relation and in 2003 revealed an alternative relation for error and disturbance to be proven universally valid. Here, we report an experimental test of Ozawa's relation for a single-photon polarization qubit, exploiting a more general class of quantum measurements than the class of projective measurements. The test is carried out by linear optical devices and realizes an indirect measurement model that breaks Heisenberg's relation throughout the range of our experimental parameter and yet validates Ozawa's relation.

Outline

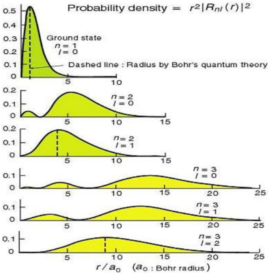
1. *Introduction*
   - Some history
   - Why do we need ITUR?
   - Rényi's entropy
2. *Entropy power UR*
   - Entropy power
3. *Applications in QM*

Why do we need ITUR?

Q: Why do we need **i**nformation-**t**heoretic **UR** in the first place?

A: Essence of **VUR** is to put an upper bound to the degree of concentration of two (or more) probability distributions ⇔ impose a lower bound to the associated uncertainties. Usual **VUR** has many limitations*:

- **variance** as a measure of concentration is a dubious concept when PDF contains more than one peak, e.g., PDF of electron in **H atom**



I. Białynicky-Birula, 1975; D. Deutsch, 1983; H. Maasen, 1988; J. Uffink, 1990

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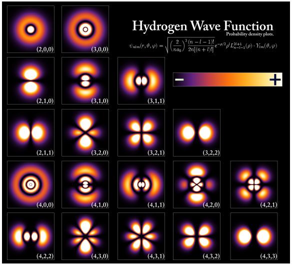

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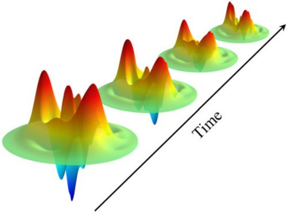


* I. Białynicky-Birula, 1975; D. Deutsch, 1983; H. Maasen, 1988; J. Uffink, 1990

Why do we need ITUR? Example I

- When the distribution is multimodal the **variance** is often non-intuitive quantifier of uncertainty

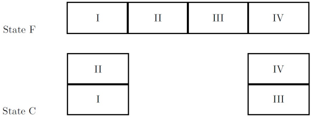
Example I: consider two states of a particle in one dimension

- First state describes a particle with a *uniform* probability density in a box of total length L, i.e.

$$\varrho \;=\; \begin{cases} 1/L, & \text{inside the box;} \\ 0, & \text{outside the box,.} \end{cases}$$

- Second state describes a particle localized with *equal* probability densities in two boxes each of length L/4,

$$\varrho \;=\; \begin{cases} 2/L, & \text{inside the box;} \\ 0, & \text{outside the box,.} \end{cases}$$



states F = flat and C = clustered

Q: In which case, F or C, is the uncertainty in the position greater?

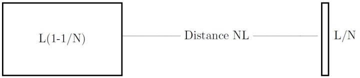
A: Intuition $\Rightarrow$ the uncertainty is greater in the case F. In the case C we know more about the position; the particle is not in the regions II and III. However, $\Delta x_F = L/\sqrt{12}$ while $\Delta x_C = \sqrt{7/4}L/\sqrt{12}$

Why do we need ITUR? Example II

- When the distribution is multimodal the **variance** does not give a sensible measure of uncertainty

Example II: consider a particle in one dimension where the probability density is *constant* in two regions I and II separated by a large distance NL (N is a large number). The region I has the size $L(1 - 1/N)$ and the distant region II has the size L/N. *Probability density* is:

$$\varrho = \begin{cases} 1/L, & \text{in region } I; \\ 1/L, & \text{in region } II; \\ 0, & \text{otherwise.} \end{cases}$$



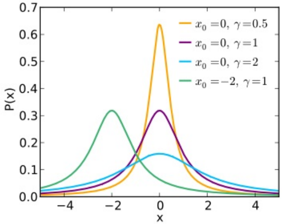
Example II: $\Delta x \sim L/\sqrt{12}\sqrt{1+12N}$.

NOTE 1: Δx tends to infinity with N even though the probability of finding the particle in the region I tends to 1

NOTE 2: Problem with the standard deviation It gets very high contributions from distant regions because these enter with a *large weight*: namely, the distance from the mean value.

Why do we need ITUR?

- **variance** diverges in many distributions even though such distributions are sharply peaked — heavy-tail distributions, e.g., Lévy, Cauchy, etc.*



Cauchy–Lorentz PDF can be freely concentrated into an arbitrarily small region by changing its scale parameter, while its standard deviation remains very large or even **infinite**.

It is desirable to quantify the inherent quantum unpredictability also in a different way, e.g., in terms of various information measures —**entropies**

*F. Lillo and R.N. Mantegna, Phys. Rev. Lett. **84** (2000).

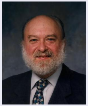
Outline

Rényi vs. Shannon — discrete case

Rényi entropy:

A. Rényi
(1921-1970)

$$\mathcal{I}_q(\mathcal{P}) \;=\; \frac{1}{1-q}\log_2\left(\sum_x p^q(x)\right), \quad q \;>\; 0$$



L.P. Kadanoff
(1937 - 2015)

- for $q = 1$ Rényi entropy equals Shannon's entropy
- is additive, i.e., $\mathcal{I}_q(\mathcal{A}_1 \cup \mathcal{A}_2) = \mathcal{I}_q(\mathcal{A}_1) + \mathcal{I}_q(\mathcal{A}_2|\mathcal{A}_1)$
- $\max_{\mathcal{P}} \mathcal{I}_q(\mathcal{P}) \;\;\Rightarrow\;\; \mathcal{P} = \{1/n, \ldots, 1/n\}$
- second law of "thermodynamics": $\mathcal{I}_q(\mathcal{B}|\mathcal{A}) \leq \mathcal{I}_q(\mathcal{B})$
- it has operational meaning via coding theorem (Campbell, 1965)

A. Rényi, 1970, 1976; L.P. Kadanoff et all, Phys. Rev. Let. 55 (1985) 2798

Rényi vs. Shannon — discrete case

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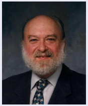

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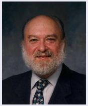

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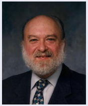

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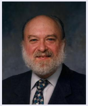

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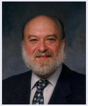

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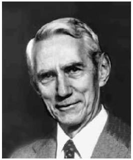
Outline

Entropy power — Shannon's case

Let $\mathcal{X}$ is a random vector in $\mathbb{R}^D$ with PDF $\mathcal{F}$. The differential (or continuous) entropy $\mathcal{H}(\mathcal{X})$ of $\mathcal{X}$ is defined as

$$\mathcal{H}(\mathcal{X}) \;=\; -\int_{\mathbb{R}^D} \mathcal{F}(\boldsymbol{x}) \log_2 \mathcal{F}(\boldsymbol{x})\, d\boldsymbol{x}$$

NOTE: Discrete version is nothing but Shannon's entropy which represents an average number of binary questions needed to reveal the value of $\mathcal{X}$.



C.E. Shannon (1916 - 2001)

NOTE: Strictly $\mathcal{H}(\mathcal{X})$ is **not** a proper entropy but rather an **information gain***.

* C.E. Shannon 1948, A. Rényi, 1970, 1976

Entropy power — Shannon's case

Entropy power $N(\mathcal{X})$ of $\mathcal{X}$ is a unique number such that*

$$\mathcal{H}(\mathcal{X}) = \mathcal{H}(\mathcal{X}_G)$$

where $\mathcal{X}_G$ is a Gaussian random vector with zero mean and variance equal to $N(\mathcal{X})$. So, equivalently

$$\mathcal{H}(\mathcal{X}) = \mathcal{H}\left(\sqrt{N(\mathcal{X})} \cdot \mathcal{Z}_G\right)$$

with $\mathcal{Z}_G$ representing a Gaussian random vector with **zero mean** and **unit covariance matrix**. The solution is* (for Shannon measured in *nats*)

$$N(\mathcal{X}) = \frac{1}{2\pi e} \exp\left(\frac{2}{D}\mathcal{H}(\mathcal{X})\right)$$

* C. Shannon 1948, M.H.M. Costa 1985

Entropy power — Rényi's case

Differential Rényi entropy $\mathcal{I}_p(\mathcal{X})$ of $\mathcal{X}$ has the form ($p \in \mathbb{R}$):

$$\mathcal{I}_p(\mathcal{X}) \;=\; \frac{1}{(1-p)} \log_2 \left(\int_M d\boldsymbol{x}\, \mathcal{F}^p(\boldsymbol{x}) \right)$$

NOTE: One can check that for $p \to 1$ one has $\mathcal{I}_p(\mathcal{X}) \to \mathcal{H}(\mathcal{X})$.

Definition

The p-th **Rényi entropy power** $N_p(\mathcal{X})$ is the solution of the equation

$$\mathcal{I}_p\left(\mathcal{X}\right) \;=\; \mathcal{I}_p\left(\sqrt{N_p(\mathcal{X})} \cdot \mathcal{Z}_G\right)$$

With $\mathcal{Z}_G$ being a Gaussian random vector with **zero mean** and **unit covariance matrix**.

Entropy power — Rényi's case

Theorem

Let $\mathcal{X}$ be a random vector in $\mathbb{R}^D$ with PDF $\mathcal{F} \in \ell^p(\mathbb{R}^D)$, where $p > 1$. The p-th **Rényi entropy power** *of $\mathcal{X}$ of the form*

$$N_p(\mathcal{X}) \quad = \quad \frac{1}{2\pi} p^{-p'/p} \exp\left(\frac{2}{D}\mathcal{I}_p(\mathcal{F})\right)$$

(with p' and p being Hölder conjugates) is the **only** *admissible class of solutions in the former equation.*

(Proof is based on the scaling property $\mathcal{I}_p(a\mathcal{X}) = \mathcal{I}_p(\mathcal{X}) + D \log |a|$ **)**

NOTE: In the limit $p \to 1_+$ one has $N_p(\mathcal{X}) \to N(\mathcal{X})$.

NOTE: There are two immediate important observations:

$$N_p(\sigma \mathcal{X}_G^{\mathbb{1}}) \ = \ \sigma^2 \quad \text{and} \quad N_p(\mathcal{X}_G^{\mathbb{K}}) \ = \ |\mathbb{K}|^{1/D} \quad (\mathcal{X}_G^{\mathbb{K}} \sim \mathcal{N}(\mathbf{0}, \mathbb{K}))$$

Entropy power uncertainty relations — B-B theorem

Theorem (Beckner–Babenko theorem)

Let
$$f^{(2)}(\boldsymbol{x}) \equiv \hat{f}^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^D} e^{2\pi i \boldsymbol{x}.\boldsymbol{y}}\, f^{(1)}(\boldsymbol{y})\, d\boldsymbol{y}$$

then for $p \in [1,2]$ *one has*

$$\|\hat{f}\|_{p'} \ \leq \ \frac{|p^{D/2}|^{1/p}}{|(p')^{D/2}|^{1/p'}}\, \|f\|_p \qquad \textit{with} \quad 1/p + 1/p' = 1$$

NOTE: Inequality is saturated only for **Gaussian** PDF's.*

Define the square-root density likelihood: $|f(\boldsymbol{y})| = \sqrt{\mathcal{F}(\boldsymbol{y})}$ then BBI implies

$$\left(\int_{\mathbb{R}^D} [\mathcal{F}^{(2)}(\boldsymbol{y})]^{(1+t)}\, d\boldsymbol{y}\right)^{1/t} \left(\int_{\mathbb{R}^D} [\mathcal{F}^{(1)}(\boldsymbol{y})]^{(1+r)}\, d\boldsymbol{y}\right)^{1/r} \leq\ [2(1+t)]^D\, |t/r|^{D/2r}$$

($r = p/2 - 1$ and $t = p'/2 - 1 \Rightarrow t = -r/(2r+1)$)

* E.H. Lieb, 1990

Entropy power uncertainty relations

When the negative logarithm is applied on both sides then

$$\mathcal{I}_{1+t}(\mathcal{F}^{(2)}) + \mathcal{I}_{1+r}(\mathcal{F}^{(1)}) \;\geq\; \frac{1}{r}\log[2(1+r)] + \frac{1}{t}\log[2(1+t)]$$

This is equivalent to

$$N_{1+t}(\mathcal{F}^{(2)})N_{1+r}(\mathcal{F}^{(1)}) \;=\; N_{p/2}(\mathcal{X})N_{q/2}(\mathcal{Y}) \;\geq\; \frac{1}{16\pi^2}$$

NOTE 1: When both $\mathcal{X}$ and $\mathcal{Y}$ represent random **Gaussian** vectors then

$$|\mathbb{K}_{\mathcal{X}}|^{1/D}|\mathbb{K}_{\mathcal{Y}}|^{1/D} \;=\; \frac{1}{16\pi^2}$$

NOTE 2: When $\mathcal{X}$ ia random vector with the covariant matrix $(\mathbb{K}_{\mathcal{X}})_{ij}$ then

$$N(\mathcal{X}) \;\leq\; |\mathbb{K}_{\mathcal{X}}|^{1/D} \;\leq\; \sigma^2_{\mathcal{X}}$$

Enters QM

Consider state vectors that are Fourier transform duals, i.e.

$$\psi(\mathbf{x}) = \int_{\mathbb{R}^D} e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}\,\hat{\psi}(\mathbf{p})\,\frac{d\mathbf{p}}{(2\pi\hbar)^{D/2}}\,,$$

$$\hat{\psi}(\mathbf{p}) = \int_{\mathbb{R}^D} e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}\,\psi(\mathbf{x})\,\frac{d\mathbf{x}}{(2\pi\hbar)^{D/2}}$$

Comparing with entropy power UR's we have

$$f^{(2)}(\mathbf{x}/\sqrt{2\pi\hbar}) = (2\pi\hbar)^{D/4}\psi(\mathbf{x})\,,$$

$$f^{(1)}(\mathbf{p}/\sqrt{2\pi\hbar}) = (2\pi\hbar)^{D/4}\hat{\psi}(\mathbf{p})$$

Consequently we can write the associated RE-based UR's as

$$N_{1+t}(|\psi|^2)N_{1+r}(|\hat{\psi}|^2) \geq \frac{\hbar^2}{4}$$

Reconstruction theorem

NOTE: In the case that the PDFs are Gaussian, the whole family of **REPURs** reduces to the single familiar **VUR**

$$\sigma_x^2 \sigma_p^2 = \frac{\hbar^2}{4}$$

Q: In what sense is the entire **tower** of REPURs more general than a single Robertson–Schrödinger **VUR**?

A: ↘ ↘ ↘

Theorem

In order to uniquely reconstruct the underlying PDF for observed QM system one needs to know ***all*** *associated entropy powers* *.

NOTE: In cases when the underlying distribution has **all cumulants** finite ⇔ Hamburger–Stiltjes moment problem

* PJ., J. Dunningham and J. Joo, AOP 2014; PRE 2016

Reconstruction theorem

RE can be written as

$$\mathcal{I}_p(\mathcal{X}) = \frac{1}{(1-p)} \log_2 \mathbb{E}\left[2^{(1-p)i_{\mathcal{X}}}\right]$$

Here $i_{\mathcal{X}}(\boldsymbol{x}) \equiv -\log_2 \mathcal{F}(\boldsymbol{x})$ is the information in $\boldsymbol{x}$.

$\Rightarrow$ RE can be viewed as a reparametrized version of the cumulant generating function of the variable $i_{\mathcal{X}}(\mathcal{X}) \Rightarrow$ cumulant expansion

$$p\mathcal{I}_{1-p}(\mathcal{X}) = \log_2 e \sum_{n=1}^{\infty} \frac{\kappa_n(\mathcal{X})}{n!} \left(\frac{p}{\log_2 e}\right)^n$$

$\kappa_n(\mathcal{X}) \equiv \kappa_n(i_{\mathcal{X}})$ is the n-th cumulant of $i_{\mathcal{X}}(\mathcal{X}) \Rightarrow$ reconstruction theorem $\Rightarrow$

- Gaussian PDF is the only PDF that saturates all REPURs.
- When $N_{1/2}(\mathcal{F}^{(1)})N_{\infty}(\mathcal{F}^{(2)}) = \hbar^2/4$ or $N_{1/2}(\mathcal{F}^{(2)})N_{\infty}(\mathcal{F}^{(1)}) = \hbar^2/4$ then the respective peak-tail parts are Gaussian.
- The closer is $\mathcal{F}$ to Gaussian the smaller neighbourhood of $p = 1$ is needed in N_p.

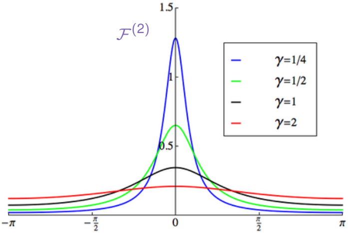
Simple examples I — heavy tailed distributions

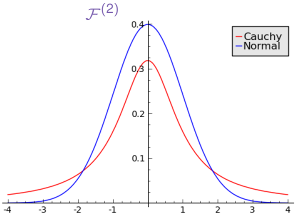
- Consider the wave function

$$\psi(x) \;=\; \sqrt{\frac{\gamma}{\pi}}\sqrt{\frac{1}{\gamma^2+(x-m)^2}} \;\;\Rightarrow\;\; \hat{\psi}(p) \;=\; e^{-i\gamma p/\hbar}\sqrt{\frac{2\gamma}{\pi^2\hbar}}\,K_0(\gamma|p|/\hbar) \quad (\text{both} \in \ell^2(\mathbb{R}))$$

- The corresponding PDFs read

$$\mathcal{F}^{(2)}(x) \;=\; \frac{\gamma}{\pi}\,\frac{1}{\gamma^2+(x-m)^2}\,, \qquad \mathcal{F}^{(1)}(p) \;=\; \frac{2\gamma}{\pi^2\hbar}\,K_0^2(\gamma|p|/\hbar)$$





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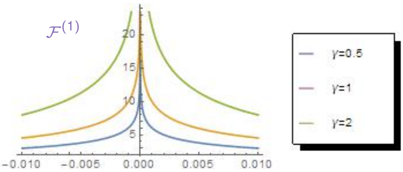

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- Particularly interesting REPURs are

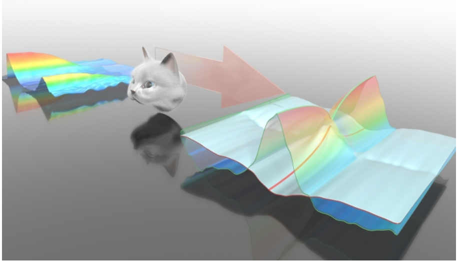
$$N_1(\mathcal{F}^{(1)})N_1(\mathcal{F}^{(2)}) = 0.0052\,\hbar^2\pi^4 > \frac{\hbar^2}{4}, \quad N_{1/2}(\mathcal{F}^{(1)})N_\infty(\mathcal{F}^{(2)}) \overset{!}{=} \frac{\hbar^2}{4}$$

cf. with $\langle(\Delta x)^2\rangle_\psi = \infty$ and $\langle(\Delta p)^2\rangle_\psi = \hbar^2\pi/16c^2$

$\Rightarrow$ Schrödinger–Robertson's VUR is completely uninformative

Simple examples II — cat states

Consider a superposition of a vacuum $|0\rangle$ and a squeezed vacuum $|z\rangle$ – **cat state**



Spectroscopy with cat states of laser light is used in material science (The Cundiff group and Brad Baxley, 2014)

Simple examples II — cat states

Consider a superposition of a vacuum $|0\rangle$ and a squeezed vacuum $|z\rangle$, i.e.

$$|\psi\rangle \ = \ \mathcal{N}\left(|0\rangle + |z_\zeta\rangle\right)$$

where

$$|z_\zeta\rangle \ = \ \sum_{m=0}^{\infty}(-1)^m \frac{\sqrt{(2m)!}}{2^m m!}\left[\frac{(\tanh\zeta)^m}{\sqrt{\cosh\zeta}}\right]|2m\rangle$$

is a superposition of even number states $|2m\rangle$ with the **squeezing** parameter ζ. $\Rightarrow$

$$\mathcal{F}^{(2)}(x) \ = \ \mathcal{N}^2\sqrt{\frac{\omega}{\pi\hbar}}\left|\exp\left(-\frac{\omega x^2}{2\hbar}\right) + e^{\zeta/2}\exp\left(-\frac{\omega e^{2\zeta}x^2}{2\hbar}\right)\right|^2$$

$$\mathcal{F}^{(1)}(p) \ = \ \mathcal{N}^2\frac{1}{\sqrt{\pi\hbar\omega}}\left|\exp\left(-\frac{p^2}{2\hbar\omega}\right) + e^{-\zeta/2}\exp\left(-\frac{e^{-2\zeta}p^2}{2\hbar\omega}\right)\right|^2$$

$$\Rightarrow \quad N_{1/2}(\mathcal{F}^{(2)})N_{\infty}(\mathcal{F}^{(1)}) \ = \ N_{\infty}(\mathcal{F}^{(2)})N_{1/2}(\mathcal{F}^{(1)}) \ = \ \frac{\hbar^2}{4}$$

Simple examples II — cat states

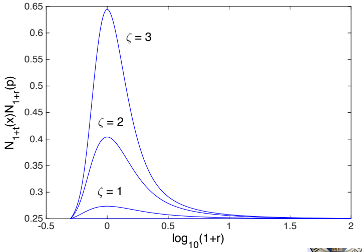
Q: What is so special about the extremal values $\mathcal{I}_{1/2}$ and $\mathcal{I}_\infty$

A: $\searrow \searrow \searrow$

Theorem

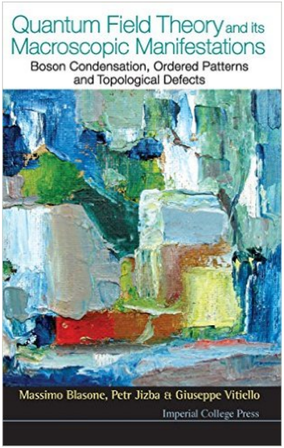
***Non-linear** nature of RE emphasizes the **more probable** parts of the PDF (typically the **middle parts**) for $p > 1$ while for $p < 1$ the **less probable** parts of the PDF (typically the **tails**) are accentuated. In other words, $\mathcal{I}_{1/2}$ mainly carries information on the rare events while $\mathcal{I}_\infty$ on the common events.*

REPUR is saturated at extremal p's because PDFs are **Gaussian** both at wings and at peaks $p = x = 0$. **REPURs** with different indices do not saturate bound.



Simple examples II — cat states

For, some further reading on (Schrödinger) cat states in condensed matter physics, see, e.g.

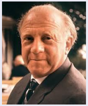


Little speculation at the end ...

NOTE: There are **information-theoretic** derivations of **black-hole** evap. formula *.

The idea is to combine Landauer principle + Heisenberg's UR

Mass temperature relation for (large) BHs



W. Heisenberg
(1901-1976)

$$m = \frac{1}{4\pi\Theta} \quad \text{with} \quad \Theta = T/T_p, \ m = M/M_p$$

R. Landauer
(1927 - 1999)

Q: What happens with the BH evaporation formula when Heisenberg's **UR** is augmented with other (higher order) **REPURs**?

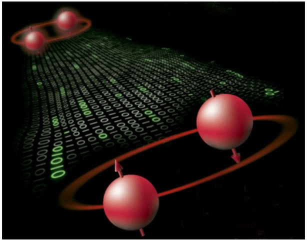
A: Either IT derivations are **hoax**, or the BH radiation spectrum gets more texture than the simple Planck's BB formula suggests.

* L. Susskind, JHEP 2005; R.J. Adler, GRG 2001; PJ., H. Kleinert and F. Scardigli, PRD 2008 · · ·

Summary

- We have generalized Shannon's ITUR to account for generalized information measures of Rényi. We have seen that in QM systems REPUR's provide more structural information on quantum states (related PDF) than conventional VUR's.

- Our method holds future promise precisely because a large part of the structure of QM is concerned with information.



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- We have generalized Shannon's ITUR to account for generalized information measures of Rényi. We have seen that in QM systems REPUR's provide more structural information on quantum states (related PDF) than conventional VUR's.
- Our method holds future promise precisely because a large part of the structure of QM is concerned with information.
- Entropy-power inequality is instrumental in treatments of QM systems with heavy tailed or multi-peak distributions (Bright–Wigner systems, Schrödinger cat states, etc.)*

* PJ, J.Dunningham and J.Joo, AOP 2015; PRE 2016

Epilogue

"Its all quite elementary, my dear Watson"

- Holmes, *A Study in Scarlet*

Arthur C. Doyle

Existent ITUR's and QM

- Landau–Pollack ineq. $\Rightarrow$ Shannon's ITUR for discrete PDF's

$$\mathcal{S}(\mathcal{P}^{(2)}) + \mathcal{S}(\mathcal{P}^{(1)}) \ \geq \ -2\log_2 c$$

- Riesz–Thorin ineq. $\Rightarrow$ Rényi's ITUR for discrete PDF's

$$\mathcal{I}_{1+t}(\mathcal{P}^{(2)}) + \mathcal{I}_{1+r}(\mathcal{P}^{(1)}) \ \geq \ -2\log_2 c \ ^{*} \quad [\smile]$$

- Beckner–Babenko ineq. $\Rightarrow$ Rényi's ITUR for continuous PDF's

$$\mathcal{N}_{p/2}(\mathcal{X})\mathcal{N}_{q/2}(\mathcal{Y}) \ \geq \ \frac{1}{16\pi^2} \ ^{*} \quad [\smile]$$

Moral

- right-hand sides are independent of the sate $|\psi\rangle$
- often more stringent bound on concentrations of PDF's than VUR's

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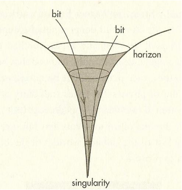
Bekenstein and Landauer

$m - \Theta$ relation for (micro) BH sensitively depends on the form of $E - x$ UP $\Rightarrow$ one can easily arrive at non-trivial phenom. consequences

Consider ensemble of unpolarized photons that deliver to MB one single bit of information per particle



In order to ensure that each photon delivers only one bit of information its position uncertainty must be of order $R_S \quad \Rightarrow \quad \Delta X_\epsilon \simeq \mu 2 R_S$. An extra bit of information added to the micro black hole will increase its energy *at least* by amount ΔE_ϵ so that

$$\Delta X_\epsilon \Delta E_\epsilon \ \simeq \ \frac{\hbar c}{2}\left[1 - \frac{\epsilon^2}{2\hbar^2 c^2}\,(\Delta E_\epsilon)^2\right]$$

Bekenstein and Landauer

With Planck's energy

$$\mathcal{E}_p = \frac{\hbar c}{2\ell_p} \approx 0.61 \cdot 10^{19}\,\mathrm{GeV}$$

GUP can be cast to

$$\Delta X_\epsilon \simeq \frac{\hbar c}{2E_\epsilon} - \frac{a^2 \ell_p E_\epsilon}{8\mathcal{E}_p} \qquad (\epsilon = a\ell_p)$$

Using the fact that, $R_S = \ell_p\, m$, where $m = M/M_p$, we can write

$$2m\mu \simeq \frac{\mathcal{E}_p}{E_\epsilon} - \frac{a^2}{8}\frac{E_\epsilon}{\mathcal{E}_p}$$

Landauer principle

Landauer principle:

When a single bit of information is erased the amount of energy dissipated into environment is *at least* $k_B T \ln 2$ where T is the temperature of erasing environment.

Liberated energy per bit of lost information cannot be grater than E_ϵ of the carrier photon

$$\Rightarrow \quad E_\epsilon \simeq k_B T \, .$$

Defining $T_p = 2\mathcal{E}_p/k_B \approx 10^{32}$ K and $\Theta = T/T_p$, we can rewrite $m - \Theta$ formula as

$$2m = \frac{1}{2\pi\Theta} - 2\pi\zeta^2\Theta$$

where $\zeta = a/(2\sqrt{2}\pi)$ and $\mu = \pi$, in order to agree with Hawking's formula in continuum limit.