No Hair Theorems and Hairy Black Holes

Federico Masella

University of Cagliari

Overview

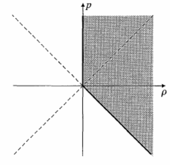
1. Introduction
   - Einstein's Equation and Energy Condition
   - General Proprieties of Black Holes
   - Schwarzschild solution
   - Kerr Solution
2. No Hair Theorems
   - Motivations
   - Uniqueness Theorems
   - Electro-vacuum vs. scalar-vacuum
   - No-scalar-hair theorems
3. Black Holes with scalar hair
   - HBH in non minimally coupled theories
   - HBH with complex scalar field
   - HBH with generic form of the potential
4. Conclusions

Einstein's Equation and Energy Condition

Einstein's Equations can be derived from a variational principle:

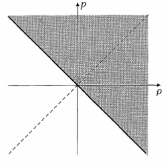
$$S = \int d^4x \sqrt{-g}\left(\frac{R}{16\pi G} + \mathcal{L}_M\right) \longrightarrow \boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}}$$

where $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$ is the **energy-momentum tensor**.



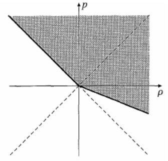
(a) WEC

Weak Energy Condition: $T_{\mu\nu}t^\mu t^\nu \geq 0$ $\Rightarrow$

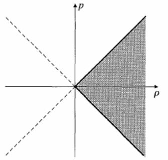


(b) NEC

Null Energy Condition: $T_{\mu\nu}\ell^\mu \ell^\nu \geq 0$



Strong Energy Condition: $T_{\mu\nu}t^\mu t^\nu \geq \frac{1}{2}T^\lambda_\lambda t^\sigma t_\sigma$

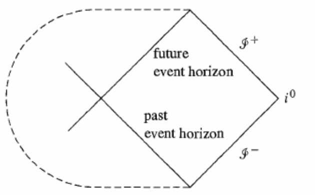


Dominant Energy Condition: $T_{\mu\nu}t^\mu t^\nu \geq 0$
$T_{\mu\nu}T^\nu_\lambda t^\mu t^\lambda \leq 0$

General Proprieties of Black Holes

A vector field generating an isometry is a **Killing vector**:

$$\nabla_{(\mu}K_{\nu)} = 0$$



1. **Event Horizon**: A null hypersurface , where $g^{rr}(r_H) = 0$
2. **Killing Horizon**: A hypersurface where a Killing vector becomes null $K^\mu K_\mu = 0$
3. **Stationary Limit Surface**: The hypersurface where the timelike Killing vector becomes null.

Schwarzschild solution (1)

- A metric with a Killing vector that is timelike near infinity is called **stationary**
- If the Killing vector is orthogonal to a family of hypersurfaces, the metric is called **static**

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Theorem (Birkhoff)

Schwarzschild metric is the unique vacuum solution with spherical symmetry.

Schwarzschild solution (2)

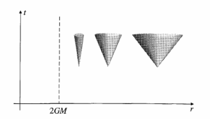


Figure: Schwarzschild coordinates

$\longrightarrow$

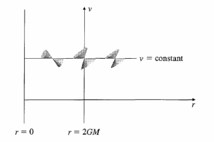
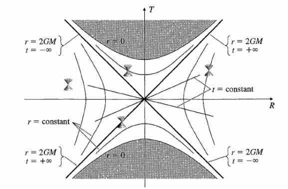
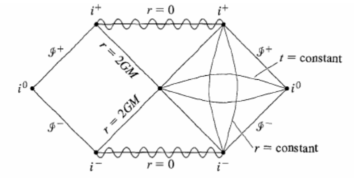


Figure: Eddington-Finkelstein coordinates

Figure: Kruskal coordinates



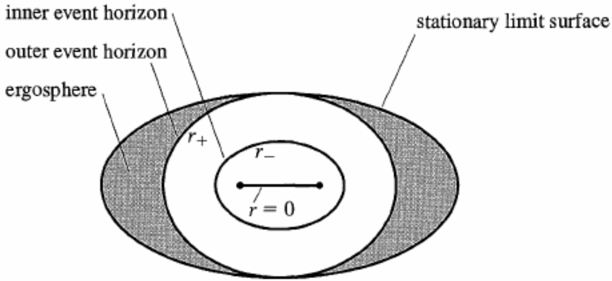


Kerr Solution

In **Boyer-Lindquist coordinates**:

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt)$$
$$+\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$$

where $\Delta(r) = r^2 - 2GMr + a^2$ and $\rho^2(r,\theta) = r^2 + a^2\cos^2\theta$



Motivations

- Uniqueness of solutions;
- Black Hole Thermodynamics;
- Stability of solutions;
- Black Hole mimics;
- GW or EM test in astrophysical environments.

THE ASTROPHYSICAL JOURNAL, 718:446–454, 2010 July 20 doi:10.1088/0004-637X/718/1/446

TESTING THE NO-HAIR THEOREM WITH OBSERVATIONS IN THE ELECTROMAGNETIC SPECTRUM.
II. BLACK HOLE IMAGES

TIM JOHANNSEN[1] AND DIMITRIOS PSALTIS[2]
[1] Physics Department, University of Arizona, 1118 E. 4th Street, Tucson, AZ 85721, USA; timj@physics.arizona.edu
[2] Astronomy Department, University of Arizona, 933 N. Cherry Ave., Tucson, AZ 85721, USA; dpsaltis@email.arizona.edu
Received 2010 February 27; accepted 2010 May 29; published 2010 June 30

IOP Publishing Classical and Quantum Gravity
Class. Quantum Grav. **33** (2016) 174001 (40pp) doi:10.1088/0264-9381/33/17/174001

Testing the black hole 'no-hair' hypothesis

Vitor Cardoso[1,2,3,5] **and Leonardo Gualtieri**[4]

Uniqueness Theorem [Israel - 1967]

For a general static metric:

$$ds^2 = -V^2(x^i)dt^2 + h_{ij}(x^i)dx^i dx^j \longrightarrow \begin{cases} h^{ij}\tilde{\nabla}_i\tilde{\nabla}_j V = 0 \\ \tilde{\nabla}_i\tilde{\nabla}_j V - \tilde{R}_{ij} V = 0 \end{cases}$$

Assumptions on the spacetime:

1. Asymptotic flatness $\Rightarrow$ The space Σ is asymptotically Euclidean;
2. It has an event horizon $\Rightarrow$ V has zeros on $\Sigma \mid V(x^i) = 0$ is a connected regular smooth surface;
3. No singularities on or outside the event horizon $\Rightarrow$ $\mathcal{R}^2 = R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ is finite everywhere on Σ for $0 \leq V < 1$

Theorem (Uniqueness for static black holes)

Any static solution of Einstein's vacuum equations satisfying conditions (1)-(3) is spherically symmetric and coincides with the Schwarzschild metric

Uniqueness and beyond

Stationary, analytic, connected, mean non-degenerate, I^+-regular electrovacuum black-hole spacetime

[asymptotically time-like Killing field k^μ]

Smoothness and topology theorems

STRONG RIGIDITY THM (1st part)

Event horizon = Killing horizon $H[\xi]$

[null generator Killing field ξ^μ]

$H[\xi]$ non-rotating: $k^\mu k_\mu|_{H[\xi]} \equiv 0$

STATICITY THM

d.o.c. static

[$k_{[\alpha}\nabla_\beta k_{\gamma]} = 0$]

STATICITY THM (2nd part)

d.o.c. static and strictly stationary

[$k^\mu k_\mu < 0$]
$\exists$ coordinate t: $k^\mu = \partial_t$ is hypersurface orthogonal

STATIC UNIQUENESS THM

[originally by means of Israel's thm,
later by the positive energy thm]

Schwarzschild (Reissner–Nordström or MP)

$H[\xi]$ rotating: $k^\mu k_\mu|_{H[\xi]} \not\equiv 0$

STRONG RIGIDITY THM (2nd part)

d.o.c. axisymmetric

[$\exists$ Killing field m^μ]

CIRCULARITY THM

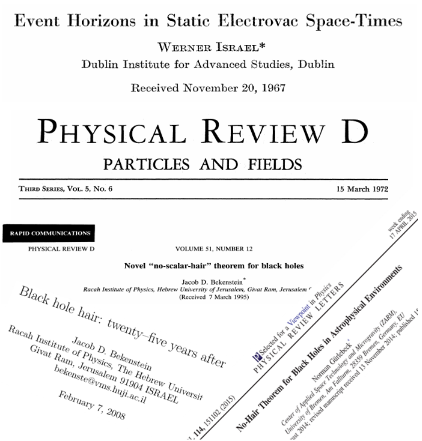
d.o.c. circular

$\exists$ coordinates t, φ: $k^\mu = \partial_t$ and
$m^\mu = \partial_\varphi$ are hypersurface orthogonal

CIRCULAR UNIQUENESS THM

[originally by means of Robinson's thm,
later by σ-model/harmonic map identities
or by inverse scattering techniques]

Kerr (Kerr–Newman) metric



Electro-vacuum vs. scalar-vacuum

$$\mathcal{L}_E = \frac{R}{4} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

- Electric field described by the **gauge potential** $A = \phi(r) dt$

$$\nabla_\mu F^{\mu\nu} = 0 \Rightarrow \phi_E(r) = -\frac{Q_E}{r}$$

$$T^E_{\mu\nu} = F_{\mu\alpha} F^\alpha_\nu - \frac{1}{4} g_{\mu\nu} F^2$$

$$(T^E)^{(\mu)}_{(\mu)} = \mp \frac{Q_E^2}{2r^4} \rightarrow \textbf{R-N Black Hole}$$

$$Q_E = \frac{1}{8\pi} \oint_{\partial\Sigma} F^{\mu\nu} dS_{\mu\nu}$$

$$\mathcal{L}_S = \frac{R}{4} - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi$$

- Scalar field described by the radial profile $\Phi(r)$

$$\Box \Phi(r) = 0 \Rightarrow \Phi = \frac{Q_S}{2M} \ln\left(\frac{2M}{r} - 1\right)$$

$$T^S_{\mu\nu} = \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} (\nabla \Phi)^2$$

$$(T^S)^{(\mu)}_{(\mu)} = \mp \frac{Q_S^2}{2r^4} \left(1 - \frac{r_H}{r}\right)^{-1} \rightarrow \varnothing$$

$$\Phi(R) = \frac{Q_S}{2M\mu} \ln\left[\frac{R - M(\mu - 1)}{R + M(\mu + 1)}\right]$$

No-scalar-hair theorem [Bekenstein -1972]

1. **Canonical and minimally coupled scalar field**

$$\mathcal{L} = \frac{R}{4} - \frac{1}{2}\nabla_\mu \Phi \nabla^\mu \Phi - V(\Phi) \;\Rightarrow\; \Box\Phi - \frac{dV}{d\Phi} = 0$$

2. **Scalar field inherits the spacetime symmetries**

$$\partial_t \Phi = 0 = \partial_\phi \Phi \;\Rightarrow\; \int d^4x\sqrt{-g}\left(\nabla_\mu \Phi \nabla^\mu \Phi + \Phi\frac{dV}{d\Phi}\right) = 0$$

3. **The potential V obeys**:

$$\Phi\frac{dV}{d\Phi} \geq 0 \;\Rightarrow\; \Phi = const.$$

Non minimally coupled or non canonical theories

1. Non minimally coupled theories *[Hawking - 1972]*

Example (Scalar-tensor theories)

Brans-Dicke theoy:

$$\mathcal{L}^{J}_{BD} = \frac{1}{16\pi}\left(\varphi\tilde{R} - \frac{\omega_0}{\varphi}\tilde{\nabla}_\mu\varphi\tilde{\nabla}^\mu\varphi\right) + \mathcal{L}_m(\tilde{g}_{\mu\nu}, \Psi_m)$$

$$\mathcal{L}^{E}_{BD} = \frac{R}{4} - \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi + 4\pi\mathcal{L}_m\left(\frac{g_{\mu\nu}}{\varphi}, \Psi_m\right)$$

Sotiriou-Faraoni theory:

$$\mathcal{L}^{J}_{ST} = \mathcal{L}^{J}_{BD} + \frac{U(\varphi)}{16\pi} \;\rightarrow\; \mathcal{L}^{E}_{ST} = \frac{1}{4\pi}\left(\mathcal{L}^{E}_{BD} + V(\Phi)\right)$$

$$\int d^4x\sqrt{-g}\left[\frac{d^2V}{d\Phi^2}\nabla_\mu\Phi\nabla^\mu\Phi + \left(\frac{dV}{d\Phi}\right)^2\right] = 0 \;\Rightarrow\; \boxed{\frac{d^2V}{d\Phi^2} \geq 0}$$

2. Non canonical scalar field theories *[Hui and Nicolis - 2013]*

Example (Galileon theories)

$\mathcal{L} = \mathcal{L}_2(X,\Phi) + \mathcal{L}_3(X,\Phi,\Box\Phi) + \;\cdots$ where $X = -\frac{1}{2}(\partial\Phi)^2 \;\rightarrow\; J^R = 0 \;\Rightarrow\; \Phi = const.$

From old to novel No-Hair Theorems

1. We start with the assumption on the potential
$$\Phi \frac{dV}{d\Phi} \geq 0$$
2. Violation of assumption 1 need **convex potential energy**:
$$\frac{d^2 V}{d\Phi^2} \geq 0$$
According to scaling techniques any hairy Black Hole must satisfy the virial relation:
$$\int_{r_H}^{\infty} dr \sigma(r) \left[\left(\frac{2r_H}{r} \left(1 - \frac{m}{r}\right) - 1 \right) \frac{1}{2} r^2 \left(\frac{d\Phi}{dr} \right)^2 + \left(\frac{2r_H}{r} - 3 \right) r^2 V(\Phi) \right] = 0$$
3. *[Heusler - 1992]* The **potential energy is non-negative (SEC)**: $V(\Phi) \geq 0$
4. *[Bekenstein - 1995]* **Energy density non-negative (WEC)**: $\rho = \frac{1}{2}(\partial\Phi)^2 + V \geq 0$

More generic scalar fields

Energy-momentum tensor must share the symmetries of the geometry:

1. *[Pena and Sudarsky - 1997]* Complex scalar field with harmonic time dependence: $\Phi \sim e^{-i\omega t}$ ruled out assuming minimally coupling, WEC and $P_r \geq P_t$
2. *[Graham and Jha - 2014]* Theories with:

$$\mathcal{L} = \frac{R}{4} + P(\Phi, X)$$

Assuming the NEC (so rigidity theorem holds), using Einstein equations and $R_{tr} = 0 = R_{t\theta}$

$$\partial_t \Phi \partial_r \Phi = 0 = \partial_t \Phi \partial_\theta \Phi \;\Rightarrow\; P = P(X) \;\Rightarrow\; \Phi = \alpha t + \beta$$

No-Hair in asymptotic AdS spacetimes

PHYSICAL REVIEW D, VOLUME 64, 044007

Scalar hair on the black hole in asymptotically anti–de Sitter spacetime

Takashi Torii*
Research Center for the Early Universe, University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan
and Advanced Research Institute for Science and Engineering, Waseda University, Shinjuku-ku, Tokyo 169-8555, Japan

Kengo Maeda†
Department of Physics, Waseda University, Shinjuku-ku, Tokyo 169-8555, Japan

Makoto Narita‡
Department of Physics, Rikkyo University, Toshima-ku, Tokyo 171-8501, Japan
(Received 9 January 2001; published 23 July 2001)

PHYSICAL REVIEW D **74,** 084008 (2006)

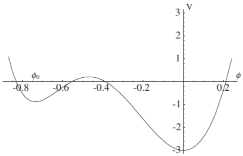
Towards a novel no-hair theorem for black holes

Thomas Hertog*
Theory Division, CERN, CH-1211 Geneva 23, Switzerland and APC, 11 Place Marcelin Berthelot, 75005 Paris, France†
(Received 17 August 2006; published 6 October 2006)

- AdS/CFT corrispondence: holography and scalar condensate;
- Hairy Black Hole solutions stabilized;
- Scalar field may have negative m^2, without destablizing the AdS vacuum, provided m^2 is above the Breitenlohner-Freedman (BF) bound.

Potential energy on the verge of violating the **Positive Energy Theorem**

THOMAS HERTOG



Conformal scalar-vacuum

$$\mathcal{L} = \frac{R}{4} - \frac{1}{2}\nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{12} R\Phi^2$$

One find the **Bocharova-Bronnikov-Melnikov-Bekenstein (BBMB) Black Hole Solution** *[1970]*:

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2$$

with secondary hair $\Phi = \frac{\sqrt{3}M}{r-M}$
This is related to scalar-tensor theories:

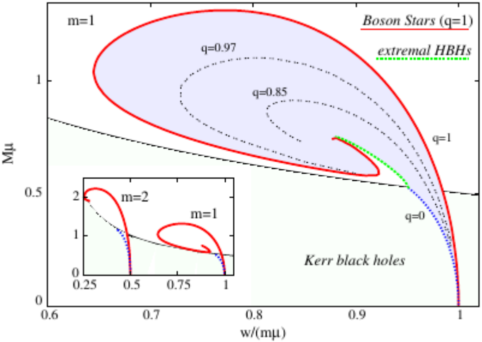
$$\varphi = 1 - \frac{1}{3}\Phi^2 \;\Rightarrow\; \varphi = \frac{r(r-2M)}{(r-M)^2}$$

Kerr BH with scalar hair [Herdeiro and Radu - 2014]

$$\mathcal{L} = \frac{R}{4} - \nabla_\mu \Phi^* \nabla^\mu \Phi - \mu^2 \Phi^* \Phi \ \rightarrow \ \Box \Phi = \mu^2 \Phi$$

$$\Phi = e^{-iwt} e^{im\phi} S_{\ell m}(\theta) R_{\ell m}(r)$$

Superraradiance threshold: $\omega = m\Omega_H$ $\rightarrow$ **bound states** $\rightarrow$ **scalar clouds**



A general method for solving the field equations

- Several solutions was found by a *scalar potential engineering*;
- *[Cadoni, Serra, Mignemi - 2011]* Reversing the usual method one can solve field equations with an ansatz for the radial profile of the scalar field.

$$S = \int d^{d+2}x\sqrt{-g}\left(R - 2(\partial\Phi)^2 - F^2 - V(\Phi)\right)$$

$$ds^2 = U(r)dt^2 + \frac{dr^2}{U(r)} + R^2(r)d\Omega^2_{(\epsilon,d)} \;\rightarrow\; Y' + Y^2 = -\frac{2}{d}\left(\Phi'\right)^2$$

Cadoni and Franzin - 2015

Hairy Black Hole solution for $\Phi \sim \frac{1}{r}$ with JNWW solution as extremal limit

- We're looking for AF solutions with spherical horizons

Conclusions

- Black Holes solutions with hair can be found in several ways;
- Do these solutions possesses primary hairs?
- Are these solutions stable?
- Is the dynamical spirit of the no-hair conjecture been falsified?
- Is any of these models phenomenologically interesting?