Relativistic symmetries and deformation

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Introduction: the (Galilean) principle of relativity

- We can characterize the symmetries of a physical system by the group of transformations that leave invariant its laws of dynamics

  All observers connected by that set of transformations describe the laws of dynamics in the same form; they describe the same physical laws
  (in a physical jargon) the laws of motion are covariant under the action of those transformations

  This defines the class of inertial observers

  For instance in special relativity the inertial observers are the class of observers connected by the Poincaré transformations, and describe the same laws of (special relativistic) dynamics

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- Galilean relativity is the relativistic framework in which Newtonian mechanics takes place

  (Galilean) principle of relativity: The laws of (Newtonian) dynamics are the same for all inertial observers (connected by the Galilei transformations)

  In Galilei relativity there is no observer-independent scale. The dispersion relation is written as $E = p^2/(2m)$ (whose structure fulfills the requirements of dimensional analysis without the need for dimensionful coefficients), and is covariant under the Galilei group of transformations

Introduction: special relativity

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- Einstein's Special Relativity introduced the first observer-independent relativistic scale (the velocity scale c), its dispersion relation takes the form $E^2 = c^2p^2 + c^4m^2$ (in which c plays a crucial role for what concerns dimensional analysis), and the presence of c in Maxwell's equations is now understood not as a manifestation of the existence of a preferred class of inertial observers but as a manifestation of the necessity to deform the Galilei transformations

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- The Galilei transformations would not leave invariant the relation $E^2 = c^2p^2 + c^4m^2$, which is instead covariant according to the Lorentz transformations (a dimensionful deformation of the Galilei transformations)

  Lorentz-Poincaré (in special relativity) transformations, enforce covariance of Maxwell equations of motion, so that the velocity “c” of light is the same for all inertial observers (without the need for an ether).

Introduction: Maximally symmetric spaces → de Sitter

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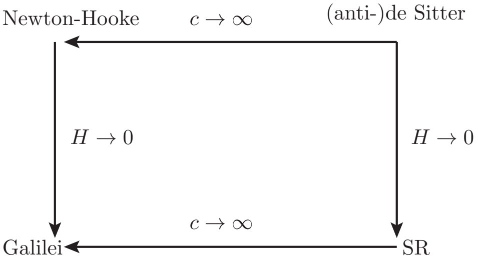
Maximally symmetric spacetimes are homogeneous and isotropic. The most general of these are de Sitter (and anti-de Sitter) spacetimes. The others can be considered as specific limits (contractions) of these (Bacry+Lévy-Leblond,1968)



de Sitter spacetime is a solution of FRW equations describing an accelerating empty universe with cosmological constant Λ. It can be considered a deformation of special relativity in terms of a time scale $H^{-1} = c/(\sqrt{\Lambda/3})$

(I will not consider anti-de Sitter)

Outline

1. Galilean relativity in covariant Hamiltonian formalism
   - Covariant Hamiltonian formalism
   - Galilean relativity
2. Special relativity as a deformation of Galileian relativity
   - Poincaré algebra
   - Relative rest and relative simultaneity
   - Loss of simultaneity and synchronization of clocks
3. de Sitter relativity
   - de Sitter particle in covariant Hamiltonian formalism
   - Non-commutativity of translations
   - Redshift as relative locality in momentum space
4. DSR theories
   - DSR example: κ-Poincaré
   - Relative locality: an insight
   - ”lateshift“ (time-delay)
5. Outlook

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Hamiltonian system ← phase space (cotangent bundle $T^\star Q$) ≡ positions and momenta ← symplectic structure

bilinear form (Poisson bivector)

$$\Omega = \left(\begin{array}{c|c} \{p_\mu, p_\nu\} & \{p_\mu, x_\nu\} \\ \hline \{x_\mu, p_\nu\} & \{x_\mu, x_\nu\} \end{array}\right)$$

Poisson brackets

$$\{f(k), g(k)\} = \Omega_{ab}(k) \frac{\partial f(k)}{\partial k_a} \frac{\partial g(k)}{\partial k_b}$$

$$\Omega_{canonical} = \left(\begin{array}{c|c} 0 & \eta \\ \hline -\eta & 0 \end{array}\right) \qquad \{f(k), g(k)\} = \frac{\partial f(k)}{\partial p_\mu} \frac{\partial g(k)}{\partial x^\mu} \qquad \eta = \mathrm{diag}(1,-1,-1,-1)$$

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Hamiltonian vector field $X_f = \frac{d}{ds} = \{f(k), \cdot\}$ → symplectic transformation (preserves the symplectic structure)

Any $f(k) = \mathcal{H}$ can be used as Hamiltonian, and its flow determines the equations of motion, as evolution in terms of the parameter τ:

Hamiltonian flow $$\frac{d}{d\tau} = \{\mathcal{H}, \cdot\} \ .$$ (Ballentine(1998))

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$$\{\mathcal{H}, G\} = 0 \quad \Rightarrow \quad \frac{dG}{d\tau} = \{\mathcal{H}, G\} = 0$$

$$\Updownarrow$$

$$\{G, \mathcal{H}\} = 0 \quad \Rightarrow \quad \partial\mathcal{H} = \mathcal{H}(k + \delta k) - \mathcal{H}(k) = \epsilon\{G, \mathcal{H}\} = 0$$

Noether theorem: the constants of motion, i.e. the conserved quantities, are the generating functions of those infinitesimal symplectic transformations that leave the Hamiltonian invariant, i.e. of the symmetry transformations (under which the equations of motion are covariant)

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Finite transformations

$$k(s) = k_0 + s\{G, k\}\Big|_0 + \frac{s^2}{2!}\{G, \{G, k\}\}\Big|_0 + \frac{s^3}{3!}\{G, \{G, \{G, k\}\}\}\Big|_0 + \cdots = \exp(sG) \triangleright k \qquad \text{(Lie algebra} \rightarrow \text{Lie group)}$$

Galilean relativity

Galilean group → Lie algebra (leaves invariant the metrics $g_{\mu\nu} = \mathrm{diag}\,(1, 0, 0, 0)$ $g^{\mu\nu} = \mathrm{diag}\,(0, 1, 1, 1)$) (central extension $\mathcal{G} \times \langle m \rangle$)
(Ardenghi+Castagnino+Campoamor-Stursberg(2009))

$$\left\{p_j, p_k\right\} = 0, \qquad \left\{p_{0,}p_j\right\} = 0,$$

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Casimir/Hamiltonian constraint → physical motion → "on-shell relation" (w is the "internal energy")

$$\mathcal{H} = C - mw = m p_0 - \frac{\vec{p}^2}{2} - mw \,. \qquad \xrightarrow{\mathcal{H}\to 0} \qquad p_0\left(p_j\right) = \frac{\vec{p}^2}{2m} + w \qquad (p_0 = E)$$

Covariant (constrained) Hamiltonian system, the motion emerges as the unfolding of a Gauge transformation, time and space are treated more symmetrically (Henneaux)

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phase space:

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Eq. of motion:

$$\dot{x}_0 = \frac{dx_0}{d\tau} = \{\mathcal{H}, x_0\} = m$$

$$\dot{x}_j = \frac{dx_j}{d\tau} = \left\{\mathcal{H}, x_j\right\} = p_j$$

$$\Rightarrow \qquad x_j\,(x_0)_{p,m} = \bar{x}_j + \frac{p_j}{m}\,(x_0 - \bar{x}_0)$$

velocity of a free particle

$$\vec{v}\,(\vec{p}) = \frac{\partial \vec{x}(x_0)}{\partial x_0} = \left.\frac{\dot{\vec{x}}}{\dot{x}_0}\right|_{\mathcal{H}=0} = \frac{\partial p_0(\vec{p})}{\partial \vec{p}} = \frac{\vec{p}}{m}$$

Galilean relativity

Rotations: $\left\{R_j, R_k\right\} = \epsilon_{jkl} R_l \quad \longrightarrow \quad \left[R_j, R_k\right] = \epsilon_{jkl} R_l$ SU(2) (or SO(3))

$$\exp\left(\vec{\alpha} \cdot \vec{R}\right) \equiv \exp\left(i\vec{\alpha} \cdot \vec{\sigma}\right)$$
$$= \cos\left(\frac{\alpha}{2}\right)\mathbb{1} + i \sin\left(\frac{\alpha}{2}\right)\hat{\alpha} \cdot \sigma$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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$\rightarrow$ summation law of angles $\quad \exp(\alpha^j R_j)\exp(\beta^j R_j) = \exp((\alpha\oplus\beta)^j R_j)$ (Baker-Campbell-Hausdorff)

$$(\alpha\oplus\beta)_j = \frac{2\cos^{-1}\left(\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\hat{\alpha}\cdot\hat{\beta}\right)}{\sin\cos^{-1}\left(\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right) - \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\hat{\alpha}\cdot\hat{\beta}\right)}$$
$$\times\left(\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\hat{\beta}_j + \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\hat{\alpha}_j - \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\left(\hat{\alpha}\wedge\hat{\beta}\right)_j\right)$$

Notice that it is non-commutative but associative:
$$\alpha\oplus\beta \neq \beta\oplus\alpha$$
$$(\alpha\oplus\beta)\oplus\gamma = \alpha\oplus(\beta\oplus\gamma)$$

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Galilean boost: $\exp\left(\vec{\xi}\cdot\vec{N}\right) \qquad [N_j, N_k] = 0$ (abelian)

summation law of velocities is obviously linear:
$$\xi\oplus\chi = \xi+\chi$$
$$u\oplus v = u+v$$
$(\vec{v} = \vec{\xi})$

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Special relativity

c^{-1} deformation of (extended) Galilei algebra

$$\left\{p_j, p_k\right\} = 0\,, \qquad \left\{p_0, p_j\right\} = 0\,,$$

$$\left\{R_j, R_k\right\} = \epsilon_{jkl} R_l\,, \qquad \left\{R_j, p_0\right\} = 0\,, \qquad \left\{R_j, p_k\right\} = \epsilon_{jkl} p_l$$

$$\left\{N_j^G, N_k^G\right\} = -\frac{1}{c^2}\epsilon_{jkl} R_l, \quad \left\{R_j, N_k^G\right\} = \epsilon_{jkl} N_l^G, \quad \left\{N_j^G, p_0^G\right\} = p_j, \quad \left\{N_j^G, p_k\right\} = \delta_{jk} m + \frac{1}{c}\delta_{jk} p_0^G$$

Special relativity

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$$p_0^{SR} = \frac{1}{c} p_0^G + mc \qquad \Rightarrow \qquad \left\{N_j, p_k\right\} = \frac{1}{c}\delta_{jk} p_0^{SR}, \quad \left\{N_j, p_0^{SR}\right\} = \frac{1}{c} p_j$$

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(procedure: inverse of Inönü-Wigner contraction)

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How the Casimir changes:

$$C_G \to m p_0^G - \frac{\vec{p}^2}{2} + \frac{1}{2c^2}\left(p_0^G\right)^2 = \frac{1}{2}\left(p_0^{SR}\right)^2 - \frac{\vec{p}^2}{2} - \frac{1}{2} m^2 c^2$$
$$C_{SR} = 2C_G + m^2 c^2 = \left(p_0^{SR}\right)^2 - \vec{p}^2$$

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Poincaré Lie algebra $SO(3,1) \ltimes \mathcal{T}_4$ leaving invariant the metric $\eta = \mathrm{diag}(1,-1,-1,-1))$ (Minkowski)

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$$\longrightarrow \quad c p_0\left(\vec{p}\right) = E = c\sqrt{\vec{p}^2 + m^2 c^2}$$

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$$\dot{x}_0 = \frac{dx_0}{d\tau} = \{\mathcal{H}, x_0\} = p_0\,,$$
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Transformation laws between observers are Poincaré (Lorentz + translations)

Special relativity

Boosts are non-commutative $\left[N_j^{SR}, N_k^{SR}\right] = -\epsilon_{jkl} R_l$

$$\mathrm{SL(2,C)/SU(2)} \ni a(\xi) = e^{\xi^j N_j} = \exp\left(\tfrac{1}{2}\vec{\xi}\cdot\sigma\right) = \cosh\left(\tfrac{1}{2}\xi\right)\mathbb{1} + \sinh\left(\tfrac{1}{2}\xi\right)\hat{\xi}\cdot\sigma$$

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Maximum velocity c: $0\oplus_1 v\oplus_2\cdots\oplus_n v = c\left(1-2\frac{\left(1-\frac{v}{c}\right)^n}{\left(1-\frac{v}{c}\right)^n+\left(1+\frac{v}{c}\right)^n}\right)\xrightarrow{n\to\infty} c$

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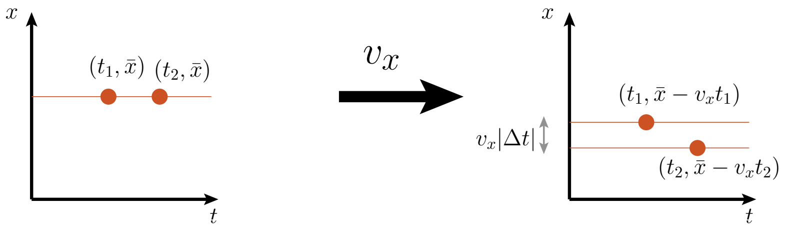
Thomas precession: $\tan\left(\tfrac{1}{2}\rho\right)\hat{\rho} = \dfrac{\tanh\left(\tfrac{1}{2}\xi\right)\tanh\left(\tfrac{1}{2}\chi\right)\hat{\xi}\wedge\hat{\chi}}{1+\tanh(\xi)\tanh(\chi)\hat{\xi}\cdot\hat{\chi}}$

Galilean relative rest

- In Galilean relativity we can say that one has relativity of ”spatial locality“



the rest is relative

We can use this scheme to describe the paradigmatic situation in which Bob is on a boat moving at velocity v_x respect to Alice, who is standing on the dock. Imagine that Alice is bouncing a ball on the dock, and that the two points mark the position and time of two of the ball's bounces on the ground. While Alice evidently observes the ball bouncing at the same point in space, Bob, who is moving with velocity v_x relative to Alice, observes, in its reference frame, the ball bouncing in two different positions: if Bob is approaching the dock, for example, Bob sees the second bounce closer then the first

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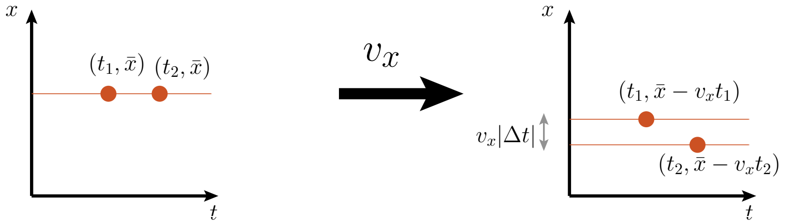

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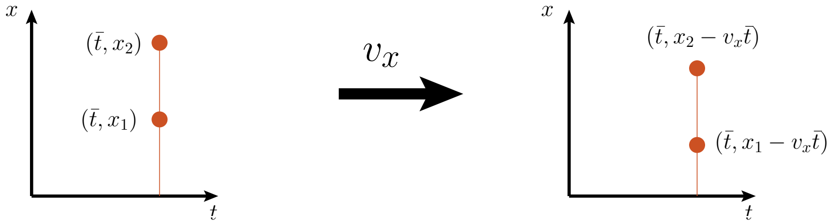
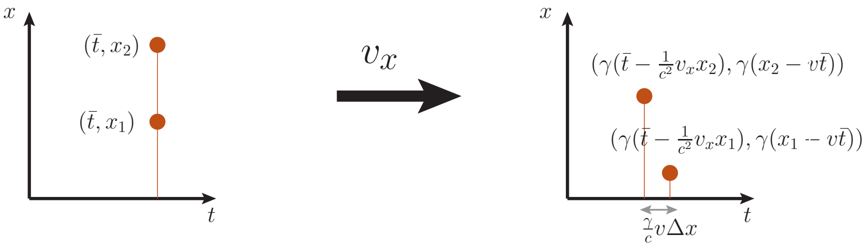


while time simultaneity is still absolute

SR: relative simultaneity

- In special relativity:

invariant scale "c" ⇒ absolute (time) simultaneity → relative (time) simultaneity.



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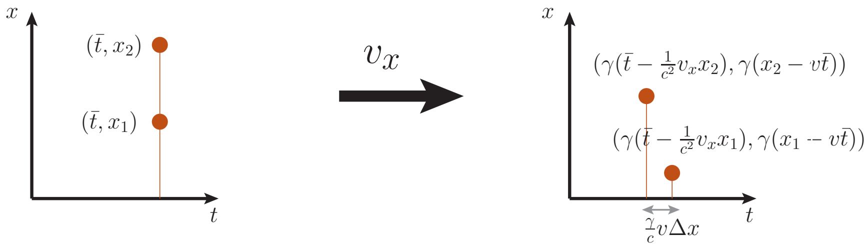
  invariant scale "c" $\Rightarrow$ absolute (time) simultaneity $\rightarrow$ relative (time) simultaneity.



- Thus one has relative space locality and relative (time) simultaneity, but still absolute spacetime locality.

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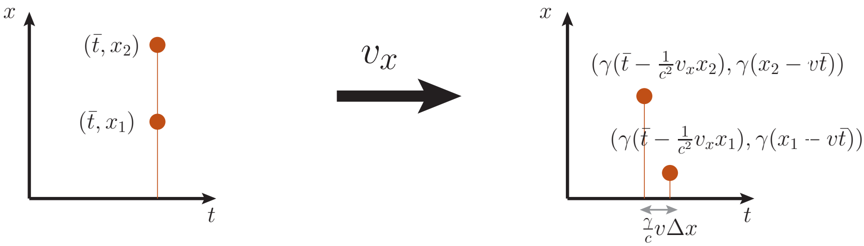


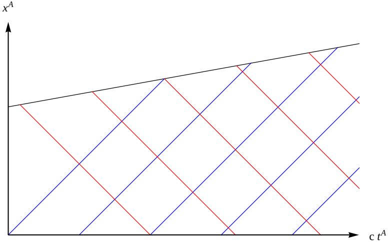
- Thus one has relative space locality and relative (time) simultaneity, but still absolute spacetime locality.
- There is no observer-independent projection from spacetime to separately space and time. We can say that one ”sees“ spacetime as a whole.

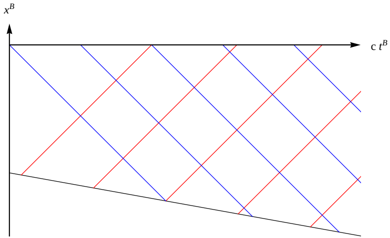
Loss of simultaneity and synchronization of clocks

Alice and Bob, distant observers in relative motion (with constant speed), have stipulated a procedure of clock synchronization and they have agreed to build emitters of blue photons (blue according to observers at rest with respect to the emitter). They also agreed to then emit such blue photons in a regular sequence, with equal time spacing Δt^*. Bob's worldlines are obtained combining a translation and a boost transformation (Bob = $\mathcal{B} \triangleright \mathcal{T} \triangleright$ Alice),

$$x_1^B\left(x_0^B\right)_p = \gamma\left(\bar{x}^A - a_1 - \beta\left(\bar{x}_0^A - a_0\right)\right) + \frac{p_1^A - \beta p_0^A}{p_0^A - \beta p_1^A}\left(x_0^B - \gamma\left(\bar{x}_0^A - a_0 - \beta\left(\bar{x}^A - a_1\right)\right)\right),$$

We arranged the starting time of each sequence of emissions so that there would be two coincidences between a detection and an emission, which are of course manifest in both coordinatizations, so to obtain a specular description. Relative simultaneity is directly or indirectly responsible for several features that would appear to be paradoxical to a Galilean observer (observer assuming absolute simultaneity). In particular, while they stipulated to build blue-photon emitters they detect red photons, and while the emissions are time-spaced by Δt^* the detections are separated by a time greater than Δt^*.





Outline

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de Sitter relativity

De Sitter spacetime is a particular case of the Friedman-Robertson-Walker (FRW) solutions of Einstein equations (with cosmological constant), in which the time dependence of the scale factor is given by the equation for the expansion rate $H = c\sqrt{\Lambda/3} \sim 10^{-19}\text{sec}^{-1}$ (in comoving time)

$$\frac{\dot{a}(t)}{a(t)} = H \qquad \text{with } H \text{ constant}$$

- De Sitter relativity can be thought of as a deformation of special relativity by the introduction of a time H^{-1} as an observer-invariant scale.
- The constancy of the expansion rate allows to define a class of inertial observers characterized by the whole set of (H-deformed) spacetime symmetries (translations, rotations and boosts), i.e. de Sitter spacetime is maximally symmetric. This is not the case for the general FRW expanding spacetime, in which the time dependence of H breaks the invariance under time translations.

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- de Sitter Lie algebra ≡ $SO(4,1)$ (Bacry+Lévy-Leblond,1968,Cacciatori+Gorini+Kamenschik,2008)

$$\{p_0, p_j\} = -\frac{H^2}{c^2}N_j\,, \qquad \{N_j, p_0\} = p_j\,, \qquad \{R_j, p_0\} = 0\,,$$

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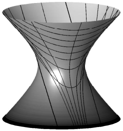

de Sitter manifold: 5D hyperboloid $X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = -\frac{c^2}{H^2}$

Flat (space slices) coordinates $ds^2 = c^2 dt^2 - a^2(t)\, d\vec{x}^2$ $a(t) = e^{Ht}$

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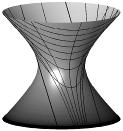

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phase space

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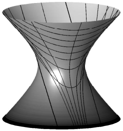

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$$C = \mathcal{P}_0^2 - e^{-2Ht}\vec{p}^2 \qquad \mathcal{H} = C - m^2 c^2$$

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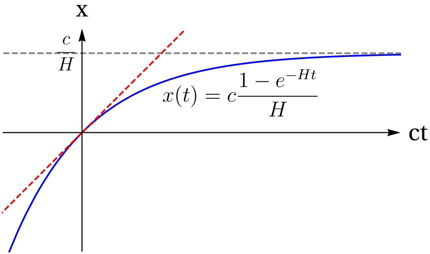
$$\vec{v} = -\frac{\partial \mathcal{P}_0(\vec{p})}{\partial \vec{p}} = \frac{\vec{p}e^{-2Ht}}{\sqrt{m^2 c^2 - e^{-2Ht}\vec{p}^2}} \quad \xrightarrow{m\to 0} \quad e^{-Ht}$$



de Sitter relativity

Momenta are non-commutative:

$$\left[p_0, p_j\right] = -\frac{H^2}{c^2} N_j, \qquad \left[p_j, p_k\right] = \frac{H^2}{c^2} \epsilon_{jkl} R_l \qquad \exp\left(a^\mu p_\mu\right) \equiv \mathrm{SO}(4,1)/\mathrm{SO}(3,1)$$

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Summation of ”position-shifts“ is non-linear

$$\left(a^{(1)} \oplus a^{(2)}\right)_\mu = \left(a_0^{(1)} + a_0^{(2)}, a_j^{(1)} + e^{-Ha_0^{(1)}} a_j^{(2)}\right)$$

$$\overrightarrow{(0 \oplus_1 a \oplus_2 \cdots \oplus_n a)} = \vec{a} \sum_{k=0}^{n} e^{-kHa_0} = \vec{a} \frac{1 - e^{-(n+1)Ha_0}}{1 - e^{-Ha_t}} \xrightarrow{n\to\infty} \frac{\vec{a}}{1 - e^{-Ha_0}}$$

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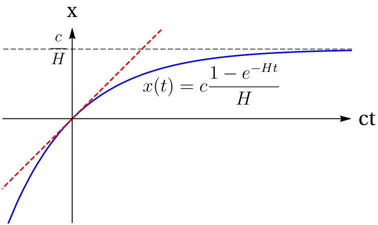
for a translation along a massless particle’s worldline



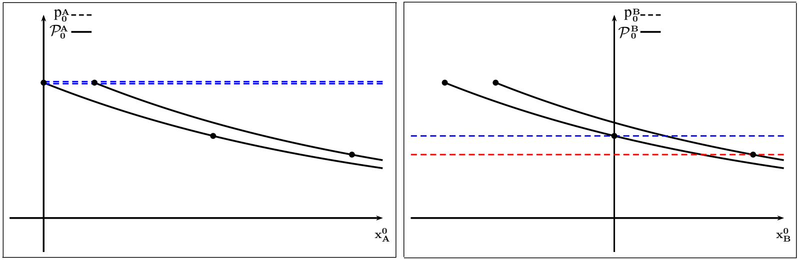
$$\Rightarrow \qquad \vec{a} = c \frac{1 - e^{-Ha_0}}{H}$$

$$\overrightarrow{(0 \oplus_1 a \oplus_2 \cdots \oplus_n a)} \xrightarrow{n \to \infty} \frac{c}{H}.$$

Cosmological horizon

Redshift as relavitve locality in momentum space

(Amelino-Camelia+Barcaroli+Gubitosi+Loret,2013)



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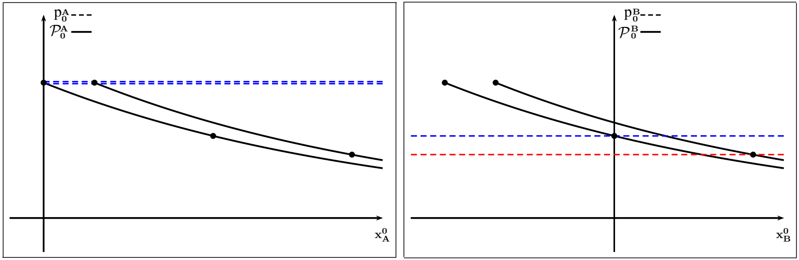

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$$\vec{p}^{\,B} = \exp\left(-Ha_0\right)\vec{p}^{\,A}$$
$$t^B = t^A - a_0$$

$$\mathcal{P}_0^B = \mathcal{P}_0^A$$

$$\Delta E_{(\mathrm{det})} = \frac{a(t_{\mathrm{em}})}{a(t_{\mathrm{det}})}\Delta E_{(\mathrm{em})} = \frac{1}{1+z}\Delta E_{(\mathrm{em})} \qquad \left(a(t) = \exp(Ht)\right)$$

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DSR theories

- Quantum gravity:

$$\text{Minimum length (Planck length)} \quad \longleftrightarrow \quad \text{Maximum energy scale}$$

(Planck scale)

$$L_p = \sqrt{\hbar G/c^3} \sim 10^{-35} m \qquad E_p = \sqrt{\hbar c^5/G} \sim 10^{19} c/\text{GeV}$$

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- DSR (Doubly Special Relativity or Deformed Relativistic Symmetries) theories where introduced to investigate the possibility of introducing, beside c, a <u>fundamental inverse-momentum scale</u> ℓ (in Quantum Gravity $\sim$ Planck scale: $\ell \sim c/E_p = \sqrt{G/(\hbar c^3)} \sim 10^{-19} c/\mathrm{GeV}$) as a relativistic <u>invariant</u>

  The requirements of DSR then are that the laws of physics involve both a fundamental velocity scale c and a fundamental inverse-momentum scale ℓ, and that each inertial observer can establish the same measurement procedure to determine the value of ℓ (besides the invariant measurement procedure to establish the value of c)

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An example: κ-Poincaré (Hopf algebra) (Lukierski,Majid,90',Amelino-Camelia,Kowalski-Glikman2000')

$$\{p_\mu, p_\nu\} = 0\,, \qquad \{R_j, R_k\} = \epsilon_{jkl} R_l\,, \qquad \{\mathcal{N}_j, \mathcal{N}_k\} = -\epsilon_{jkl} R_l\,,$$

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$$C_\ell = \left(\frac{2}{\ell}\right)^2 \sinh^2\left(\frac{\ell}{2} p_0\right) - e^{\ell p_0}\vec{p}^2 \qquad (\kappa = 1/\ell)$$

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phase space

(κ-Minkowski)

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$$\left\{p_j, x_0\right\} = -\ell p_j\,, \qquad \left\{p_j, x_k\right\} = -\delta_{jk}\,,$$

("Heisenberg principle in spacetime")

$$R_j = \epsilon_{jkl} x_k p_l\,, \qquad \mathcal{N}_j = -x_0 p_j + x_j\left(\frac{1 - e^{-2\ell p_0}}{2\ell} + \frac{\ell}{2}\vec{p}^2\right)$$

DSR example: κ-Poincaré

$$\{p_\mu, p_\nu\} = 0\,, \qquad \{R_j, R_k\} = \epsilon_{jkl} R_l\,, \qquad \{\mathcal{N}_j, \mathcal{N}_k\} = -\epsilon_{jkl} R_l\,,$$

$$\{R_j, \mathcal{N}_k\} = \epsilon_{jkl}\mathcal{N}_l\,, \qquad \{R_j, p_0\} = 0\,, \qquad \{R_j, p_k\} = \epsilon_{jkl} p_l\,,$$

$$\{\mathcal{N}_j, p_0\} = p_j\,, \qquad \{\mathcal{N}_j, p_k\} = \delta_{jk}\left(\frac{1 - e^{-2\ell p_0}}{2\ell} + \frac{\ell}{2}\vec{p}^2\right) - \ell p_j p_k\,,$$

$$C_\ell = \left(\frac{2}{\ell}\right)^2 \sinh^2\left(\frac{\ell}{2}p_0\right) - e^{\ell p_0}\vec{p}^2$$

phase space

(κ-Minkowski)

$$\{p_0, x_0\} = 1\,, \qquad \{p_0, x_j\} = 0\,, \qquad \{x_j, x_0\} = \ell x_j, \qquad \{x_j, x_k\} = 0$$

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("Heisenberg principle in spacetime“)

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Free particle

$$\mathcal{H} = C - m^2c^2 \longrightarrow p_0(\vec{p}) \xrightarrow{m\to 0} -\frac{1}{\ell}\ln\left(1 - \ell\left|\vec{p}\right|\right)$$

$$v = c\frac{\partial p_0(\vec{p})}{\partial \vec{p}} \xrightarrow{m=0} c\exp(\ell p_0)$$

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Summation of momenta:

$$[x_j, x_0] = \ell x_j \qquad \longrightarrow \qquad \exp(p_0 x_0)\exp(p_j x_j) \in \mathcal{AN}_3^\star$$

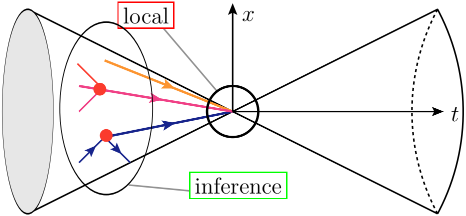
$$(p \oplus q)_\mu = \left(p_0 + q_0, p_j + e^{-\ell q_0} p_j\right)$$

Maximum energy/momentum

$$(0 \oplus_1 p \oplus_2 \cdots \oplus_n p) \xrightarrow{n\to\infty} |\vec{p}|\,\frac{1}{1 - e^{-\ell p_0}} = \frac{1}{\ell}$$

Relative locality: an insight

We don’t actually “see” spacetime, but we “see” (detect) time sequences of particles, and then abstract spacetime by inference:



We actually “see” (detect) only what we locally witness

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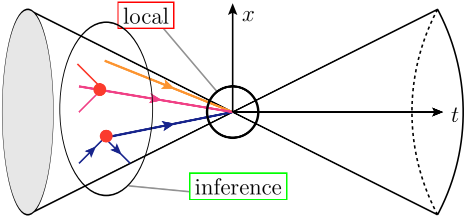
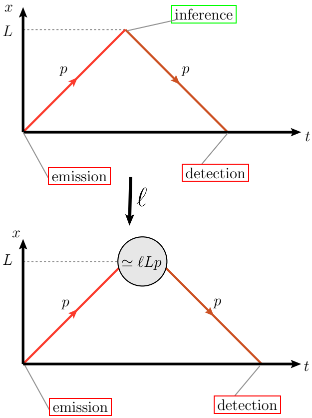


Think to the Einstein clock

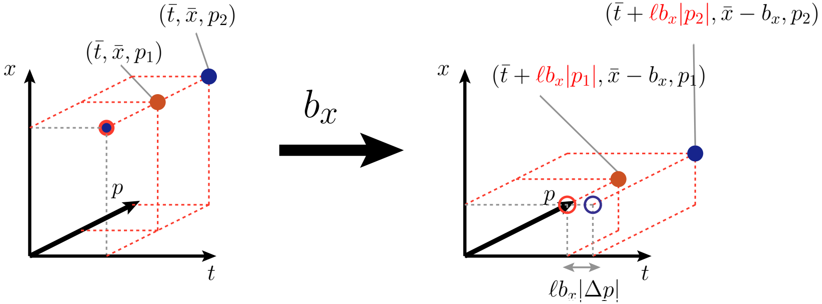


We actually "see" (detect) only what we locally witness

Relative locality: DSR theories

- DSR theories:

invariant (inverse-momentum) scale ℓ
$\Rightarrow$ absolute spacetime locality $\rightarrow$ relative spacetime locality



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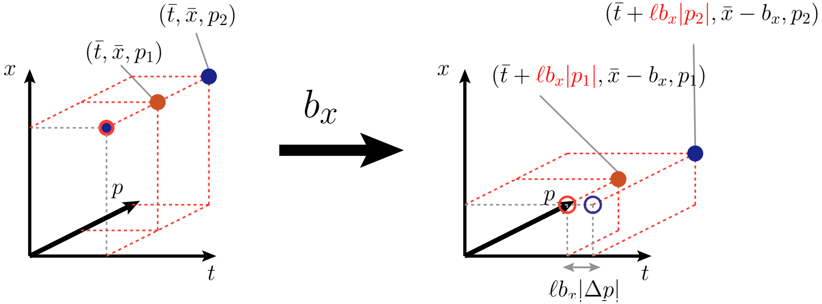
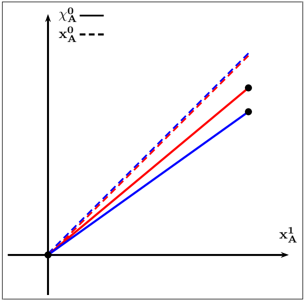
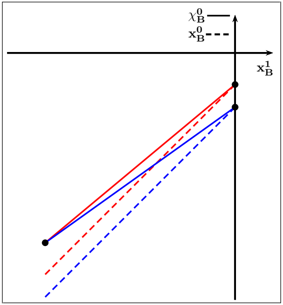


- There is no observer-independent projection from a one-particle phase space to a description of the particle separately in spacetime and in momentum space.
  We thus can say that one ”sees“ phase space as a whole.

"Lateshift" (time-delay)





$$\{p_0, x_0\} = 1 \ , \qquad \{p_0, x_j\} = 0 \ ,$$
$$\{p_j, x_0\} = -\ell p_j \ , \qquad \{p_j, x_k\} = -\delta_{jk} \ ,$$
$$\{x_j, x_0\} = \ell x_j, \qquad \{p_0, p_j\} = 0.$$

$$\xrightarrow{\chi_0 = x_0 - \ell \vec{x}\cdot\vec{p}}$$

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$$\left.\frac{d\vec{x}\,(x_0)}{dx_0}\right|_{m=0} = 1,$$

$$\left.\frac{d\vec{x}\,(\chi_0)}{d\chi_0}\right|_{m=0} = e^{-\ell p_0},$$

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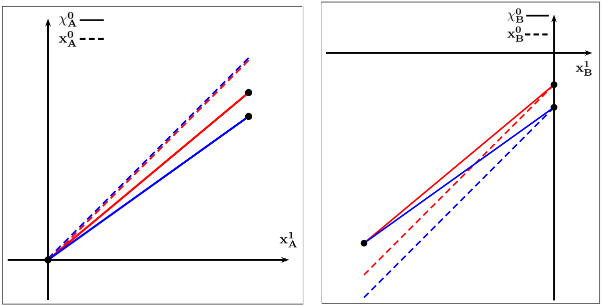

$$\{p_0, x_0\} = 1\,, \quad \{p_0, x_j\} = 0\,, \quad \{p_j, x_0\} = -\ell p_j\,, \quad \{p_j, x_k\} = -\delta_{jk}\,, \quad \{x_j, x_0\} = \ell x_j, \quad \{p_0, p_j\} = 0.$$

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$$\left.\frac{d\vec{x}(x_0)}{dx_0}\right|_{m=0} = 1, \qquad \left.\frac{d\vec{x}(\chi_0)}{d\chi_0}\right|_{m=0} = e^{-\ell p_0},$$

$$x_0^B = x_0^A - a_0 + \ell \vec{a}\cdot\vec{p} \qquad \vec{x}^B = \vec{x}^A - \vec{a} \qquad \chi_0^B = \chi_0^A - a_0$$

$$a_0 = |\vec{a}| = T, \qquad \vec{a} = |\vec{a}|\hat{p}, \qquad |\vec{p}| = \frac{1 - e^{-\ell p_0}}{\ell}$$

$$\boxed{\Delta t = \ell T \Delta |\vec{p}| = T\left(e^{-\ell p_0^s} - e^{-\ell p_0^h}\right) \sim \ell T \Delta E}$$

Outline

1. Galilean relativity in covariant Hamiltonian formalism
   - Covariant Hamiltonian formalism
   - Galilean relativity
2. Special relativity as a deformation of Galileian relativity
   - Poincaré algebra
   - Relative rest and relative simultaneity
   - Loss of simultaneity and synchronization of clocks
3. de Sitter relativity
   - de Sitter particle in covariant Hamiltonian formalism
   - Non-commutativity of translations
   - Redshift as relative locality in momentum space
4. DSR theories
   - DSR example: κ-Poincaré
   - Relative locality: an insight
   - ”lateshift“ (time-delay)
5. Outlook

Outlook

- Phenomenological opportunities: Testing Planck-scale in-vacuo dispersion relation (time delays) with gamma-ray-bursts and IceCube astrophysical neutrinos
  (Amelino-Camelia+D'Amico+Loret+G.R.,NatureAstrophysics1(2017))

$$\Delta t = \eta_X \frac{E}{E_P} D(z) \pm \delta_X \frac{E}{E_P} D(z)$$

$$D(z) = \int_0^z d\zeta \frac{(1+\zeta)}{H_0 \sqrt{\Omega_\Lambda + (1+\zeta)^3 \Omega_m}}$$

- DSR-de Sitter and DSR-FRW scenarios
  (G.R.+Amelino-Camelia+Marcianò+Matassa,PhysRevD92(2015))
- Relative locality in DSR theories
  (Amelino-Camelia+Matassa+Mercati+G.R.,PhysRevLett106(2011))
  (Amelino-Camelia+Arzano+Kowalski-Glikman+G.R.+Trevisan,ClassQuantGrav29(2012))
- Relative locality in Snyder spacetime
  (Mignemi+G.R.,(2018))