An introduction to gravitational lensing and a possible combined analysis with rotation curves

Matteo Trudu

Università degli Studi di Cagliari

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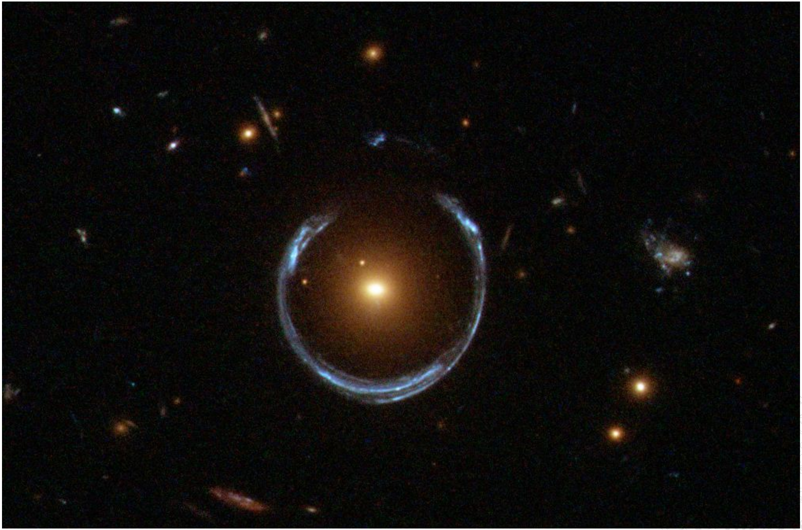
Summary

- Introduction to gravitational lensing
  1. Deflection angle
  2. General properties of a gravitational lens
  3. Lensing regimes
- Combining rotation curves and gravitational lensing

Gravitational Lensing



An observation taken with the Hubble Space Telescope

Gravitational Lensing

Historical highlights

- In 1783, speculating that light consists of corpuscoles, an astronomer, named John Michell (1724-1793) hypothesized the possibility that a gravitational field could bend a light ray. He found out the following result for the deflection angle:

$$\alpha = \frac{2GM}{c^2\xi}$$

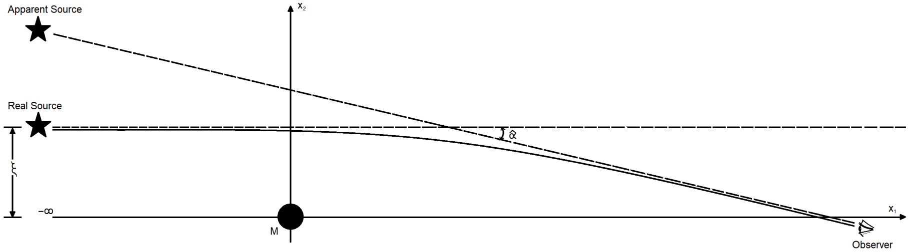
John Michell

Sir Arthur Eddington

- With the full equations of General Relativity did Einstein obtain twice the Newtonian value:

$$\alpha = \frac{4GM}{c^2\xi} = 2\frac{R_S}{\xi}$$

- Eddington used the gravitational lens of the sun to test Einstein's theory of general relativity during the solar eclipse of the 29^{th} May 1919. He measured a deflection angle of $1.75''$ which is compatible with the Einstein's prediction!



In weak field approximation:

$$ds^2 = -\left(1 + \frac{2\Phi_N}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\Phi_N}{c^2}\right) |d\vec{x}|^2$$

Geodesic equation:

$$\frac{du^\mu}{d\tau} + \Gamma_{\alpha\beta}{}^{\mu} u^\alpha u^\beta = 0$$

for a photon we have

$$Dk^\mu = dk^\mu + \Gamma_{\alpha\beta}{}^{\mu} dx^\alpha k^\beta = 0$$

we have to compute

$$dk^2 = -\Gamma_{\alpha\beta}{}^{2} dx^\alpha k^\beta$$

$$dx^\alpha = (cdt, dx^1, 0, 0) \qquad k^\beta = (\omega/c, k^1, 0, 0)$$

The Christofell's symbols:

$$\Gamma_{\alpha\beta}{}^{\mu} = \frac{1}{2}\eta^{\mu\rho}\left(\partial_\alpha h_{\rho\beta} + \partial_\beta h_{\alpha\rho} - \partial_\rho h_{\alpha\beta}\right)$$

hence

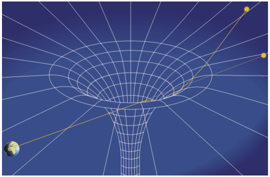
$$dk^2 = -\left(\Gamma_{00}{}^2 + 2\Gamma_{01}{}^2 + \Gamma_{11}{}^2\right)\frac{\omega}{c}dx^1 = -\frac{2\omega}{c^3}\frac{GMx_2}{\left(x_1^2 + x_2^2\right)^{3/2}}dx^1$$

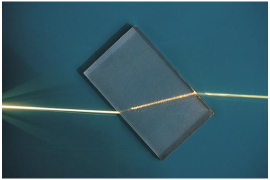
$$\frac{\Delta k^2}{k^1} \simeq -\frac{2GM\xi}{c^2}\int_{-\infty}^{\infty}\frac{1}{\left(x_1^2 + \xi^2\right)^{3/2}}dx_1 = -\frac{4GM}{c^2\xi}$$

Since the gravitational field is weak we are entitled to assume small deflection angle

$$\tan\left(\hat{\alpha}\right) \simeq \hat{\alpha} = \left|\frac{\Delta k^2}{k^1}\right| = \frac{4GM}{c^2\xi}$$

Light deflection can equivalently be described by Fermat's principle, as in geometrical optics!





Fermat's principle reads:

$$\delta \int dt = \frac{1}{c} \int n[\vec{x}(l)]dl = 0$$

We need a refractive index. Since $ds^2 = 0$ for photons the light speed in the gravitational field is thus

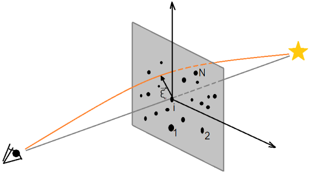
$$c' = \left| \frac{d\vec{x}}{dt} \right| = c\sqrt{\frac{1 + \frac{2\Phi_N}{c^2}}{1 - \frac{2\Phi_N}{c^2}}} \simeq c\left(1 + \frac{2\Phi_N}{c^2}\right)$$

$$n = \frac{c}{c'} = \frac{1}{\left(1 + \frac{2\Phi_N}{c^2}\right)} \simeq 1 - \frac{2\Phi_N}{c^2}$$

Fermat's principle suggests us a Lagrangian approach! Considering a generic parametric curve for the light's path one finds:

$$\boxed{\hat{\vec{\alpha}} = \frac{2}{c^2} \int_{-\infty}^{+\infty} \nabla_\perp \left(\Phi_N\right) dz}$$

Since for a point-like mass the deflection angle is proportional to the mass we can use the superposition principle.



$$\hat{\vec{\alpha}}(\vec{\xi}) = \sum_{i=1}^{N} \hat{\vec{\alpha}}_i(\vec{\xi} - \vec{\xi}_i) = \frac{4G}{c^2} \sum_{i=1}^{N} M_i \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2}$$

The typical length-scales for the lenses are low compared to the distances between observer and source!

Thin Screen Approximation
Mass projected on the line of sight:

$$\Sigma(\vec{\xi}) = \int \rho\left(\vec{\xi}, z\right) dz$$

$$\boxed{\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \Sigma(\vec{\xi}') \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} d^2\vec{\xi}'}$$

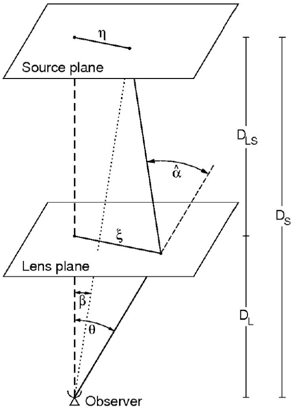


Figure from P. Schneider et al. *Gravitational Lenses*, 1992, Spinger-Verlag

$$\vec{\eta} = \vec{\beta} D_S : \quad \text{Source Position}$$
$$\vec{\xi} = \vec{\theta} D_L : \quad \text{Image Position}$$

If $\hat{\vec{\alpha}}$ is the deflection angle, by a geometrical construction we can write a relation between the angles.

Lens Equation

$$\vec{\theta} D_S = \vec{\beta} D_S + \hat{\vec{\alpha}}\left(\vec{\theta}\right) D_{LS}$$
$$\vec{\alpha}\left(\vec{\theta}\right) = \frac{D_{LS}}{D_S} \hat{\vec{\alpha}}\left(\vec{\theta}\right)$$
$$\vec{\beta} = \vec{\theta} - \vec{\alpha}\left(\vec{\theta}\right)$$

Let's introduce the dimensionless quantities $\vec{\xi} = \xi_0 \vec{x}$ and $\vec{\eta} = \eta_0 \vec{y}$:

$$\boxed{\vec{y} = \vec{x} - \vec{\alpha}\left(\vec{x}\right)}$$

An extended distribution of matter is characterized by its effective Lensing Potential

$$\Psi\left(\vec{x}\right) = \frac{2D_L D_{LS}}{c^2 \xi_0^2 D_S} \int_{-\infty}^{\infty} \Phi_N\left(\vec{x}, z\right) dz$$

This lensing potential satisfies two important properties:

1. the gradient of Ψ gives the scaled deflection angle

$$\nabla_x \Psi\left(\vec{x}\right) = \vec{\alpha}\left(\vec{x}\right)$$

2. the Laplacian of Ψ gives twice the *convergence*

$$\nabla_x^2 \Psi\left(\vec{x}\right) = 2\kappa\left(\vec{x}\right)$$

where

$$\kappa\left(\vec{x}\right) = \frac{\Sigma(\vec{x})}{\Sigma_{crit}} \qquad \Sigma_{crit} = \frac{c^2}{4\pi G} \frac{D_S}{D_L D_{LS}}$$

One of the main features of gravitational lensing is the distortion which it introduces into the shape of the sources. The distortion arises because light bundles are deflected differentially. The distortion of images can be described by the Jacobian matrix

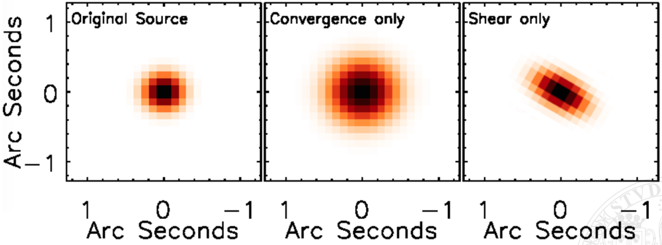
$$\mathbf{A} = \frac{\partial \vec{y}}{\partial \vec{x}} = \mathbf{I} - \mathbf{\Psi} \quad \text{where} \quad I_{ij} = \delta_{ij} \quad \Psi_{ij} = \frac{\partial^2 \Psi}{\partial x_i \partial x_j}$$

We can define the Shear Matrix:

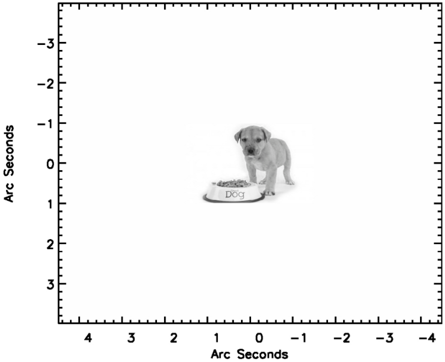
$$\begin{aligned} \mathbf{S} &= \mathbf{A} - \frac{1}{2}\mathrm{tr}\left(\mathbf{A}\right)\mathbf{I} = \begin{pmatrix} -\frac{1}{2}\left(\Psi_{11} - \Psi_{22}\right) & -\Psi_{12} \\ -\Psi_{12} & \frac{1}{2}\left(\Psi_{11} - \Psi_{22}\right) \end{pmatrix} = \begin{pmatrix} -\gamma_1 & -\gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix} \\ \vec{\gamma}(\vec{x}) &= (\gamma_1(\vec{x}), \gamma_2(\vec{x})) = \left(\frac{1}{2}\left(\Psi_{11} - \Psi_{22}\right), \Psi_{12}\right) \quad \textit{Shear Vector} \end{aligned}$$

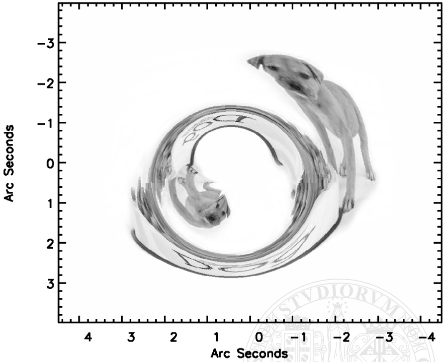
$$\boxed{\mathbf{A} = \left(1 - \kappa\left(\vec{x}\right)\right)\mathbf{I} + \mathbf{S}}$$

- **Convergence** : encodes all the variations of the surface of the source, which possess the same shape. It rescale the surface of the source. (scalar quantity)
- **Shear** : encodes all the variations of the shape of the source, which possess the same surface. (vectorial quantity)



Effect of convergence and shear on a gaussian PSF
Credit: Andrea Enia, *Ricostruzione di sorgenti gravitazionalmente lensate selezionate nel sub-millimetrico*, Master Degree Thesis





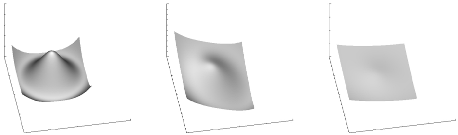
Credit: Andrea Enia, *Ricostruzione di sorgenti gravitazionalmente lensate selezionate nel sub-millimetrico*, Master Degree Thesis

Gravitational Lensing
Occurence of images (I)

The deflection of light rays causes a delay in the time between the emission of radiation by the source and the signal reception by the observer.

$$\tau(\vec{x}) = \frac{1+z_L}{c}\frac{\xi_0^2 D_S}{D_L D_{LS}}\left[\frac{1}{2}(\vec{x}-\vec{y})^2 - \Psi(\vec{x})\right]$$

this equation implies that images satisfy the Fermat Principle $\nabla_x \tau(\vec{x}) = 0$.



Credit: M.Meneghetti,*Introduction to Gravitational Lensing*

Time delay surfaces of an axially symmetric lens for three different source positions.

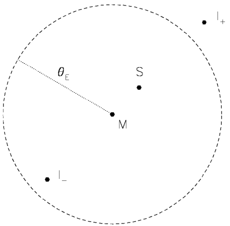
Let's consider a point-like mass acting as a lens.

- Deflection angle:

$$\alpha = \frac{4GM}{c^2 D_L \theta}$$

- Lensing Potential:

$$\Psi(\theta) = \frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}\ln|\theta|$$



The lens equation reads:

$$\beta = \theta - \frac{4GMD_{LS}}{c^2\theta D_L D_S} = \theta - \frac{\theta_E^2}{\theta} \qquad \theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_S D_L}} \quad \text{Einstein Radius}$$

solving for θ :

$$\theta^2 - \beta\theta - \theta_E^2 = 0 \qquad \text{Two images, if } \beta = 0 \rightarrow \text{Einstein Ring}$$

Simulation: Point lens and extended source

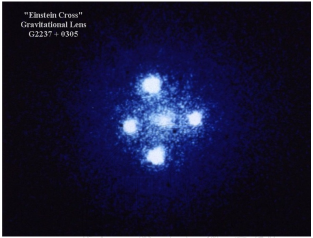
Unlensed

Lensed

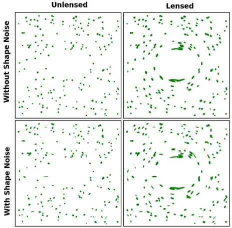
Credit: Joachim Wambsganss, *Gravitational Lensing Theory and Applications*

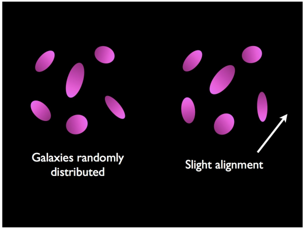
- multiple images of background sources. The displacement of such images is determined by the mass distribution of the lenses.
- highly distorted images. If the source is extended, the differential deflection of the light creates distortions in the images.

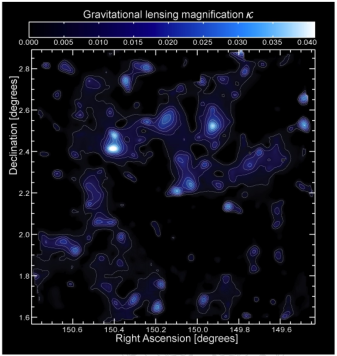




- weak distortions and small magnifications.
- we need a stastical approach.







- microlensing phenomena are produced by lenses whose sizes are small compared to the scale of the lensing system
- can be thought of as a version of strong gravitational lensing in which the image separation is too small to be resolved.
- such lenses can be for example planets, stars or any compact object floating in the halo or in the bulge of our or of other galaxies.

- Mass and mass distribution of galaxy clusters.
- Determine cosmological parameters (Hubble constant) from the time delay.
- Dark matter distribution in single galaxies.
- Map the distribution of Dark matter at cosmological scales.
- Dark matter and dark energy nature.
- Lensing as a gravitational telescope to study very faint and distant objects (e.g. Quasars).
- Discover new extrasolar planets in the Milk Way.
- Testing alternative theories of gravity.

Combining rotation curves and gravitational lensing

Tristan Faber, Matt Visser, "*Combining rotation curves and gravitational lensing: How to measure the equation of state of dark matter in the galactic halo*", (2006) [arXiv:astro-ph/0512213v2]

Credit: NASA/ESA and The Hubble Heritage Team (STScI/AURA)

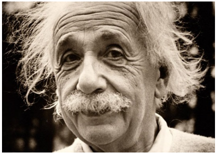
The static and approximately spherically symmetric gravitational field of a galaxy is represented by the space-time metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{1}{1 - 2m(r)/r}dr^2 + r^2 d\Omega^2$$

The most general static and spherically symmetric stress-energy tensor:

$$T_{\mu\nu} = \begin{pmatrix} -\rho(r) & 0 & 0 & 0 \\ 0 & p_r(r) & 0 & 0 \\ 0 & 0 & p_t(r) & 0 \\ 0 & 0 & 0 & p_t(r) \end{pmatrix}$$

One can find a relationship between the metric functions and the density and pressures profiles using the Einstein field equations.



Newtonian Gravity

$$\nabla^2 \Phi\left(\vec{r}\right)_N = 4\pi G \rho\left(\vec{r}\right)$$

Einstenian Gravity

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

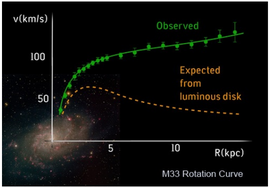
$$\text{W.F.A.} \rightarrow \nabla^2 \Phi \simeq 4\pi G \left(\rho + p_r + 2p_t\right)$$

Standard Newtonian gravity is obtained in the limit of General Relativity where

1. the gravitational field is weak (W.F.A.);
2. the probe particle speeds involved are slow compared to the speed of light;
3. the pressure and matter fluxes are small compared to the mass-energy density.

For the regime of rotation curves measurements, conditions of weak field and low speed for test particles can be applied. The geodesic equation reads:

$$\frac{d^2\vec{x}}{dt^2} \simeq -\nabla\Phi$$



For egde-on galaxies the total wavelenght shift of an emission line in W.F.A. (Nucamendi, Salgando & Sudarsky 2001)

$$1 + z_{\pm}(r) \simeq 1 \mp \sqrt{r\Phi'(r)}$$

In Newtonian Gravity $z_N^2 = r\Phi'(r)_N$

$$\text{but} \quad \Phi \neq \Phi_N \; ! \rightarrow \Phi = \Phi_{RC} m_{RC}^N = 4\pi \int \rho r^2 dr \quad m_{RC} \simeq 4\pi \int (\rho + p_r + 2p_t) r^2 dr$$

We want to characterise the entire trajectory of light rays with a single effective refractive index.

$$ds^2 = e^{2\Phi(\tilde{r})}\left(-dt^2 + n(\tilde{r})^2\left[d\tilde{r}^2 + \tilde{r}^2 d\Omega^2\right]\right)$$

In W.F.A one finds:

$$n(r) = 1 - 2\Phi_{lens}(r)$$

$$\Phi_{lens}(r) = \frac{1}{2}\Phi(r) + \frac{1}{2}\int \frac{m(r)}{r^2}dr$$

$$\nabla^2\Phi_{lens}(r) = 4\pi\rho_{lens} \quad \Phi_{lens} = \int \frac{m_{lens}(r)}{r^2}dr$$

$\Phi_{lens} \neq \Phi_N \rightarrow \rho_{lens} \neq \rho$ and $m_{lens} \neq m$! Just to sum up:

$$\boxed{\Phi_{RC}(r) = \Phi(r) \quad \Phi_{lens}(r) = \frac{1}{2}\Phi(r) + \frac{1}{2}\int \frac{m(r)}{r^2}dr}$$

The masses

$$m_{RC}(r) = r^2 \Phi'(r) \quad m_{lens}(r) = r^2 \Phi'_{lens}(r)$$

From the masses, we can infer

$$4\pi r^2 \left(p_r(r) + 2p_t(r)\right) \simeq \left(m'_{RC}(r) - m'_{lens}(r)\right)$$

If $m'_{lens}(r) = m'_{RC}(r) = m'(r)$ we find the Newtonian limit. A convenient parameter that determines a measure of the equation of state:

$$w(r) = \frac{p_r(r) + 2p_t(r)}{3\rho(r)} \simeq \frac{2}{3} \frac{m'_{RC}(r) - m'_{lens}(r)}{2m'_{lens}(r) - m'_{RC}(r)}$$

This post-Newtonian formalism requires the simultaneous measurements of pseudo-density profiles from rotation curves and gravitational lensing observation.

- We can infer them from similar galaxies (Brainerd 2004);
- Combined simultaneous measurments of rotation curves and lensing of individual galaxies (Large number of rotation curves but few strong lensing systems, Sofue & Rubin 2001, Kochanek et al. 2005)

Conclusions and outlooks

- We have discussed the basics of gravitational lensing theory, in particular:
    1. we have found a deflection angle;
    2. we have discussed the general properties of a gravitational lens, e.g. the lensing potential, the convergence and the shear;
    3. we have shown the typical lensing regimes used in astrophysics.
- We have briefly discussed the possibility of a combined analysis of rotation curves and gravitational lensing.
- **Outlooks:** try to use this approach for a model in which the phenomena attributed to dark matter are due to a radial pressure generated by the reaction of the dark energy fluid to the presence of baryonic matter.

Thank you for your attention!