The intriguing relationship between fractals and coherence*

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*G.Vitiello, Coherent states, fractals and brain waves. New Mathematics and Natural Computation 5, 245-264 (2009)

Fractals and the Fock-Bargmann representation of coherent states. Quantum Interaction. Third Int. Symposium (QI-2009). Saarbruecken, Germany, Eds. P. Bruza, D. Sofge, et al. LNAI 5494, 6 (Springer, Berlin 2009)

Fractals, coherent states and self-similarity induced noncommutative geometry. Phys. Lett. A 376, 2527 (2012)

On the Isomorphism between Dissipative Systems, Fractal Self-Similarity and Electrodynamics. Toward an Integrated Vision of Nature. Systems 2, 203 (2014).

- Discussion is limited to the self-similarity property* of deterministic† fractals and logarithmic spiral.

Self-similarity properties of fractal structures and logarithmic spiral are related to

- squeezed coherent states
- quantum dissipative dynamics
- noncommutative geometry in the plane

- Not discussing: the measure of lengths in fractals, random fractals, etc.

*the most important property of fractals! p. 150 of Peitgen, H.O., Jürgens, H., Saupe, D.: Chaos and fractals. New Frontiers of Science. Springer-Verlag, Berlin (1986)

†the ones generated iteratively according to a prescribed recipe.

Consider the example of the Koch curve (Helge von Koch, 1904) *

Notice:

Koch was searching an example of curve everywhere non-differentiable ("On a continuous curve without tangents, constructible from elementary geometry")

*Peitgen, H.O., Jürgens, H., Saupe, D.: Chaos and fractals. New Frontiers of Science. Springer-Verlag, Berlin (1986)

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Quantum Interaction. Third Int. Symposium (QI-2009). Saarbruecken, Germany, Eds. P. Bruza, D. Sofge, et al. LNAI 5494, 6 (Springer 2009)

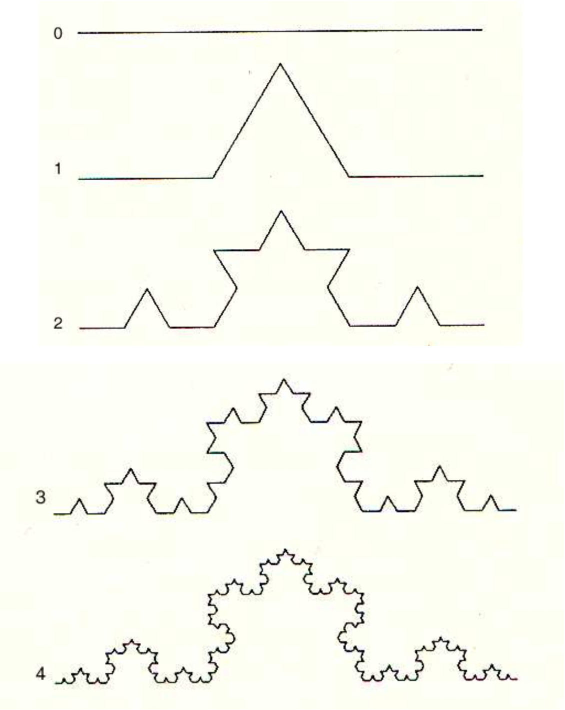


Fig. 1. The first five stages of Koch curve.

- **Stage $n = 0$:** $L_0 = u_0$ **(arbitrary initiator)** **it lives in 1 dimension**

- **Stage $n = 1$:** $u_{1,q}(\alpha) \equiv q\,\alpha\,u_0$, $q = \frac{1}{3^d}$, $\alpha = 4$ **(the generator)**

$d \neq 1$ **to be determined.** **it does not live in 1 dimension.**

The "deformation" of the u_0 segment is only possible provided the one dimensional constraint $d = 1$ is relaxed.

The u_1 segment "shape" lives in some $d \neq 1$

$d \neq 1$ **is a measure of the deformation of the dimensionality**

- **Stage $n = 2$:** $u_{2,q}(\alpha) \equiv q\,\alpha\,u_{1,q}(\alpha) = (q\,\alpha)^2\,u_0$.

- **By iteration:**

$$u_{n,q}(\alpha) \equiv (q\,\alpha)\,u_{n-1,q}(\alpha), \qquad n = 1, 2, 3, ...$$

$$u_{n,q}(\alpha) = (q\,\alpha)^n\,u_0.$$

which is the "self-similarity" relation characterizing fractals.

Notice! The fractal is mathematically defined only in the limit of infinite number of iterations ($n \to \infty$).

Normalizing, at each stage, with (arbitrary) u_0:

$$\frac{u_{n,q}(\alpha)}{u_0} = (q\,\alpha)^n = 1, \qquad \text{for each} \quad n$$

i.e.

$$d = \frac{\ln 4}{\ln 3} \approx 1.2619.$$

The non-integer d is called

fractal dimension , or self-similarity dimension .

Note that $q\,\alpha = 1$, i.e. $\frac{1}{3^d}4 = 1$ is not true for $d = 1$, i.e. if one remains in $d = 1$ dimension

The value of d, fractal dimension, is a measure of the deformation which allows to impose the “constraint” $\frac{u_{n,q}(\alpha)}{u_0} = 1 = (q\,\alpha)^n$.

Now consider in full generality the complex α-plane.

The functions

$$u_{n,q}(\alpha) = \frac{(q\,\alpha)^n}{\sqrt{n!}} \;, \qquad u_0(\alpha) = 1 \;, \qquad q = e^{d\,\theta}\,,\; \theta \in \mathbf{C} \qquad \alpha \in \mathbf{C} \;, \qquad n \in \mathcal{N}_+ \;,$$

form in the space $\mathcal{F}$ of the entire analytic functions (i.e. uniformly converging in any compact domain of the α-plane) a basis which is orthonormal.

The factor $\frac{1}{\sqrt{n!}}$ ensures the normalization condition with respect to the gaussian measure.

Consider the finite difference operator $\mathcal{D}_q$, also called the q-derivative operator:

$$\mathcal{D}_q f(\alpha) = \frac{f(q\alpha) - f(\alpha)}{(q-1)\alpha} \ ,$$

with $f(\alpha) \in \mathcal{F}$, $q = e^{\zeta}$, $\zeta \in \mathbf{C}$**.**
$\mathcal{D}_q$ **reduces to the standard derivative for $q \to 1$ ($\zeta \to 0$).**

In the space $\mathcal{F}$, the commutation relations hold:

$$[\mathcal{D}_q, \alpha] = q^{\alpha \frac{d}{d\alpha}} \quad , \quad \left[\alpha \frac{d}{d\alpha}, \mathcal{D}_q\right] = -\mathcal{D}_q \quad , \quad \left[\alpha \frac{d}{d\alpha}, \alpha\right] = \alpha \ , \tag{1}$$

which lead us to the identification

$$N \to \alpha \frac{d}{d\alpha} \quad , \quad a_q \to \alpha \quad , \quad a_q \to \mathcal{D}_q \ ,$$

with $a_q = a_{q=1} = a^\dagger$ **and** $\lim_{q\to 1} a_q = a$ **on $\mathcal{F}$.**
This algebra is the q-deformation of the WH algebra.

The operator q^N acts in the whole $\mathcal{F}$ as

$$q^N f(\alpha) = f(q\alpha) \ , \quad f(\alpha) \in \mathcal{F} \ .$$

For the coherent state functional

$$|\alpha\rangle = \exp(-\frac{|\alpha|^2}{2}) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \exp(-\frac{|\alpha|^2}{2}) \sum_{n=0}^{\infty} u_n(\alpha)|n\rangle,$$

we have[‖]

$$q^N|\alpha\rangle = |q\alpha\rangle = \exp(-\frac{|q\alpha|^2}{2}) \sum_{n=0}^{\infty} \frac{(q\alpha)^n}{\sqrt{n!}} \ |n\rangle \ ,$$

and, since $q\alpha \in \mathbf{C}$,

$$a \ |q\alpha\rangle = q\alpha \ |q\alpha\rangle \ , \quad q\alpha \in \mathbf{C} \ .$$

[‖]E. Celeghini, S. De Martino, S. De Siena, M. Rasetti and G. Vitiello, Quantum groups, coherent states, squeezing and lattice quantum mechanics, Ann. Phys. 241, 50 (1995).
E. Celeghini, M. Rasetti and G. Vitiello, On squeezing and quantum groups, Phys. Rev. Lett. 66, 2056 (1991).

Notice!

$$\frac{1}{\sqrt{n!}}(q\,\alpha)^n$$

is the "deformed" basis in $\mathcal{F}$, where coherent states are represented.

The link between fractals and coherent states is established by realizing that the fractal nth-stage function $u_{n,q}(\alpha)$, with u_0 set equal to 1, is obtained by projecting out the nth component of $|q\alpha\rangle$ and restricting to real $q\alpha$, $q\alpha \to Re(q\alpha)$:

$$\langle q\alpha|(a)^n|q\alpha\rangle = (q\alpha)^n = u_{n,q}(\alpha), \quad q\alpha \to Re(q\alpha).$$

The operator $(a)^n$ thus acts as a "magnifying" lens: the nth iteration of the fractal can be "seen" by applying $(a)^n$ to $|q\alpha\rangle$.

Note that "the fractal operator" q^N can be realized in $\mathcal{F}$ as:

$$q^N \psi(\alpha) = \frac{1}{\sqrt{q}} \psi_s(\alpha) \ ,$$

where $q = e^{\zeta}$ (for simplicity, assumed to be real) and $\psi_s(\alpha)$ denotes the squeezed states in FBR.

q^N acts in $\mathcal{F}$ as the squeezing operator $\mathcal{S}(\zeta)$ (well known in quantum optics) up to the numerical factor $\frac{1}{\sqrt{q}}$.

$\zeta = \ln q$ is called the squeezing parameter.

The q-deformation process, which we have seen is associated to the fractal generation process, is equivalent to the squeezing transformation.

These results can be extended also to the logarithmic spiral. Its defining equation in polar coordinates (r, θ) is

$$r = r_0\, e^{d\,\theta} \ , \tag{2}$$

with r_0 and d arbitrary real constants and $r_0 > 0$, whose representation is the straight line of slope d in a log-log plot with abscissa $\theta = \ln e^{\theta}$:

$$d\,\theta = \ln \frac{r}{r_0} \ . \tag{3}$$

The constancy of the angular coefficient $\tan^{-1} d$ signals the self-similarity property of the logarithmic spiral: rescaling $\theta \to n\,\theta$ affects r/r_0 by the power $(r/r_0)^n$. Thus, we may proceed again like in the Koch curve case and show the relation to squeezed coherent states.

(cf. with the Koch curve case: $(q\,\alpha)^n = 1$, with $q = e^{-d\,\theta}$, is written as $d\,\theta = \ln\,\alpha$)

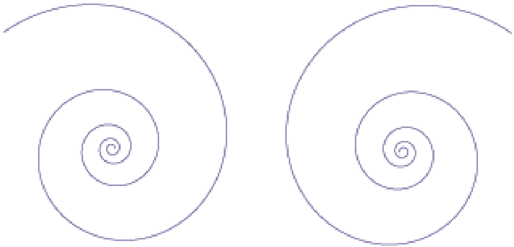


FIG. 2: The anti-clockwise and the clockwise logarithmic spiral.

The parametric equations of the spiral are:

$$x = r(\theta)\cos\theta = r_0\, e^{d\,\theta}\cos\theta\ ,$$
$$y = r(\theta)\sin\theta = r_0\, e^{d\,\theta}\sin\theta\ . \qquad (4)$$

In the complex z-plane

$$z = x + i\,y = r_0\, e^{d\,\theta}\, e^{i\,\theta}\ , \qquad (5)$$

the point z is fully specified only when the sign of $d\,\theta$ is assigned. The factor $q = e^{d\,\theta}$ may denote indeed one of the two components of the (hyperbolic) basis $\{e^{-\,d\,\theta}, e^{+\,d\,\theta}\}$.

Due to the completeness of the basis, both the factors $e^{\pm\,d\,\theta}$ must be considered.

It is interesting that in nature in many instances the direct ($q > 1$) and the indirect ($q < 1$) spirals are both realized in the same system (the most well known systems where this happens are found in phyllotaxis studies).

The points z_1 and z_2 are considered:

$$z_1 = r_0 \, e^{-d\theta} \, e^{-i\theta} \;, \qquad z_2 = r_0 \, e^{+d\theta} \, e^{+i\theta} \;, \tag{6}$$

By using the parametrization $\theta = \theta(t)$, z_1 and z_2 solve the equations

$$\begin{aligned} m\,\ddot{z}_1 + \gamma\,z_1 + \kappa\,z_1 &= 0 \;, \\ m\,\ddot{z}_2 - \gamma\,z_2 + \kappa\,z_2 &= 0 \;, \end{aligned} \tag{7}$$

respectively, provided the relation

$$\theta(t) = \frac{\gamma}{2\,m\,d}\,t = \frac{\Gamma}{d}\,t \tag{8}$$

holds (up to an arbitrary additive constant c set equal to zero). m, γ and κ positive real constants. $\Gamma \equiv \frac{\gamma}{2m}$. Then,

$$z_1(t) = r_0 \; e^{-i\,\Omega\,t} \, e^{-\Gamma\,t} \;, \qquad z_2(t) = r_0 \; e^{+i\,\Omega\,t} \, e^{+\Gamma\,t} \;, \tag{9}$$

with $\quad \Omega^2 = \frac{1}{m}(\kappa - \frac{\gamma^2}{4m}) = \frac{\Gamma^2}{d^2} \;, \qquad \kappa > \frac{\gamma^2}{4m}$ **.**

One can interpret the parameter t as the time parameter.

Time-evolution of the system of direct and indirect spirals is described by the system of damped and amplified harmonic oscillator equations.

Oscillator z_1 is an *open* non-hamiltonian system. In order to set up the canonical formalism the *closed* system (z_1, z_2), made by z_1 and its time-reversed image z_2, must be considered[**].

The "two copies" (z_1, z_2) viewed as describing the forward and the backward in time path in the phase space $\{z, p_z\}$, respectively.

As far as $z_1(t) \neq z_2(t)$ the system exhibits quantum behavior and quantum interference takes place[††]

[**] E. Celeghini, M. Rasetti and G. Vitiello, Annals of Physics(N.Y.) 215, 156 (1992).

[††] J. Schwinger, J. Math. Phys. 2, 407 (1961).
G. 't Hooft, Class. Quant. Grav. 16, 3263 (1999); J. Phys.: Conf. Series 67, 012015 (2007).
M. Blasone, P. Jizba, G. Vitiello, Phys. Lett. A 287, 205 (2001).
M. Blasone, E. Celeghini, P. Jizba, G. Vitiello, Phys. Lett. A 310, 393 (2003).

The ground state $|0(t)\rangle$ for the closed system $\{z_1, z_2\}$ is found to be an $SU(1,1)$ generalized (squeezed) coherent state.

It is a thermal state and its time evolution is controlled by the entropy operator:

$$|0(t)\rangle \;=\; \exp\left(-\frac{1}{2}\mathcal{S}_A(t)\right)\exp\left(A^\dagger B^\dagger\right)|0\rangle = \exp\left(-\frac{1}{2}\mathcal{S}_B(t)\right)\exp\left(A^\dagger B^\dagger\right)|0\rangle \;.$$

$\mathcal{S}_A(t)$ and $\mathcal{S}_B(t)$ have the same formal expressions (with B and $B^\dagger$ replacing A and $A^\dagger$):

$$\mathcal{S}_A(t) \equiv -\{A^\dagger A \ln \sinh^2(\Gamma\, t) - A A^\dagger \ln \cosh^2(\Gamma\, t)\} \;. \qquad (10)$$

Since A's and B's commute, $\mathcal{S}$ denotes either $\mathcal{S}_A$ or $\mathcal{S}_B$.

$\mathcal{S}$ is the *entropy* for the dissipative system.

Time evolution controlled by $\mathcal{S}$ $\rightarrow$ breaking of time-reversal symmetry $\rightarrow$ choice of a privileged direction in time evolution (time arrow).

The breakdown of time-reversal symmetry characteristic of dissipation is manifest in the formation process of fractals;

in the case of the logarithmic spiral the breakdown of time-reversal symmetry is associated with the chirality of the spiral: the indirect (right-handed) spiral is the time-reversed, *but distinct*, **image of the direct (left-handed) spiral.**

The Hamiltonian $\mathcal{H}$ turns out to be the fractal free energy for the coherent boson condensation process out of which the fractal is formed.

The time-evolution operator $\mathcal{U}(t)$ can be written as

$$\mathcal{U}(t) = \exp\left(-\frac{\Gamma\, t}{2}((a^2 - a^{\dagger 2}) - (b^2 - b^{\dagger 2}))\right) ,$$

in terms of a and b operators (related to A and B by a canonical transformation). $\mathcal{U}(t)$ is the two mode squeezing generator with squeezing parameter $\zeta = -\Gamma\, t$.

$|0(t)\rangle$ is thus a squeezed state.
We can repeat the construction also for the Koch curve.

For simplicity the problem has been tackled within the framework of quantum mechanics. The correct mathematical framework to study quantum dissipation is the one of quantum field theory (QFT), where one considers an infinite number of degrees of freedom.

This is also physically more realistic, because the realizations of the logarithmic spiral (and in general of fractals) in the many cases it is observed in nature involve an infinite number of elementary degrees of freedom.

The logarithmic spiral and its time-reversed double manifest themselves as macroscopic quantum systems.

Same conclusion holds for the Koch curve and other fractals.

We know that the quantum interference phase ϑ (of the Aharanov-Bohm type) between two alternative paths $\mathcal{P}_1$ and $\mathcal{P}_2$ in the plane is determined by the noncommutative deformation parameter q and the enclosed area $\mathcal{A}$: $\vartheta = \mathcal{A}/q^2$**.

In the (z_1, z_2) plane, introduce for simplicity the index notation $+ \equiv 1$ and $- \equiv 2$. The forward in time and backward in time velocities $v_\pm = z_\pm$ are given by

$$v_\pm = \frac{1}{m}\,(p_{z_\mp} \mp \frac{1}{2}\gamma z_\pm) \qquad (11)$$

and they do not commute

$$[v_+, v_-] = -i\,\frac{\gamma}{m^2}\ . \qquad (12)$$

**Sivasubramanian, S., Srivastava, Y.N., Vitiello, G., Widom, A., Phys. Lett. A 311, 97–105 (2003)
Blasone, M., Jizba, P., Vitiello, G.: Quantum Field Theory and its macroscopic manifestations. Imperial College Press, London (2011)

Define the conjugate position coordinates $\xi_{\pm} \equiv \mp(m/\gamma)v_{\pm}$, then

$$\left[\xi_{+}, \xi_{-}\right] = i\,\frac{1}{\gamma}\;, \tag{13}$$

which characterizes the noncommutative geometry in the plane $(z_{+},\, z_{-})$.

The quantum dissipative interference phase ϑ associated with the two paths $\mathcal{P}_1$ and $\mathcal{P}_2$ in the noncommutative plane is $\vartheta = \mathcal{A}\,\gamma$, provided $z_{+} \neq z_{-}$.

The “dissipative interference phase” provides a relation between dissipation and noncommutative geometry in the plane of the doubled coordinates.

Remark: the noncommutative q-deformed Hopf algebra plays a relevant rôle in the formalism of the algebra doubling.

The map $\mathcal{A} \to \mathcal{A}_1 \otimes \mathcal{A}_2$ which duplicates the algebra is the Hopf coproduct map $\mathcal{A} \to \mathcal{A} \otimes 1 + 1 \otimes \mathcal{A}$.

The deformed coproduct maps $\Delta \, a_q^\dagger = a_q^\dagger \otimes q^{1/2} + q^{-1/2} \otimes a_q^\dagger$ ($a_q^\dagger$ are the creation operators in the q-deformed Hopf algebra) are noncommutative and the q-deformation parameter is related to the coherent condensate content of the state $|0(t)\rangle$.

Energy-momentum conservation in electrodynamics

In classical electrodynamics, as well as in quantum electrodynamics:

the conservation of the electromagnetic (em) energy-momentum tensor $T^{\mu\nu}$

$$\partial_\mu T^{\mu\nu} = 0, \tag{1}$$

closed **system $\{\psi, A^\mu\}$, made of the matter field ψ and of the em gauge field A^μ.**

However, the conservation arises from the compensation between the matter part $T_m^{\mu\nu}$ and the em part $T_\gamma^{\mu\nu}$ of the total $T^{\mu\nu}$:

$$\partial_\mu T_m^{\mu\nu} = e\, F^{\alpha\nu} J_\alpha \tag{2}$$

$$\partial_\mu T_\gamma^{\mu\nu} = -e\, F^{\alpha\nu} J_\alpha \tag{3}$$

where J_α denotes the current, e is the charge and as usual $F^{\alpha\beta} = \partial^\beta A^\alpha - \partial^\alpha A^\beta$

no need to specify the boson or fermion nature of $\psi(x)$.

Volume integration gives for $\nu = 0$ the rate of changes in time of the energy of the matter field and em field, $\mathcal{E}_m$ and $\mathcal{E}_\gamma$, respectively:

$$\partial_0 \mathcal{E}_m = e\,\mathbf{E}\cdot\mathbf{v} = -\partial_0 \mathcal{E}_\gamma$$

For $\nu = i = 1, 2, 3$, integration over the volume gives

$$\partial_0 P_m^i = e\,E^i + e\,(\mathbf{v}\times\mathbf{B})^i \tag{4}$$

$$\partial_0 P_\gamma^i = -e\,E^i - e\,(\mathbf{v}\times\mathbf{B})^i \tag{5}$$

namely, the Lorentz forces F_m and F_γ, acting on two opposite charges with same velocity v in the same electric and magnetic fields, E and B, respectively.

Let, at least in some space-time region, the magnetic field B be a constant vector, thus described by the vector potential

$$\mathbf{A} = \frac{1}{2}\mathrm{B} \times \mathrm{r}$$

where $\mathrm{r} = (x_1, x_2, x_3)$. It is $\mathrm{B} = \nabla \times \mathbf{A}, \;\; \nabla \cdot \mathbf{A} = 0$.

Choose the reference frame so that $\mathrm{B} = \nabla \times \mathbf{A} = -B\hat{\mathbf{3}}$. Then, $A_3 = 0$ and by using $\epsilon_{12} = -\epsilon_{21} = 1$; $\epsilon_{ii} = 0$,

$$A_i = \frac{B}{2}\epsilon_{ij}x_j \;\; , \qquad\qquad i, j = 1, 2.$$

The third component, $i = 3$, of $(\mathrm{v} \times \mathrm{B})$ vanishes.

Assume also that E is given by the gradient of the harmonic potential $\equiv \frac{k}{2e}({x_1}^2 - {x_2}^2) \equiv \;_1 - \;_2$, $\mathrm{E} = -\nabla$; and $E_3 = 0$.

We may thus limit our analysis to the $i = 1, 2$ components.

Then let $i = 1$ and put $B \equiv \gamma/e$. We have

$$mx_1 + \gamma x_2 + kx_1 = 0$$

m, γ and k are time independent quantities. For $i = 2$:

$$mx_2 + \gamma x_1 + kx_2 = 0$$

The Hamiltonian is

$$H = H_1 - H_2 = \frac{1}{2m}(p_1 - e_1 A_1)^2 + e_1 \quad {}_1 - \frac{1}{2m}(p_2 + e_2 A_2)^2 + e_2 \quad {}_2$$

In the least energy state (where $H = 0$, $H_1 = H_2$) the respective contributions to the energy compensate each other.

One of the oscillators may be considered to represent the em field in which the other one is embedded and vice-versa.

In summary, the system of damped/amplified oscillators provides, under the conditions specified above, a representation of the content of Maxwell equations and the associated conservation laws.

The quantum field theory framework

the damped/amplified oscillators have a quantum representation in terms of squeezed $SU(1,1)$ coherent states.

Thus, the isomorphism between $SU(1,1)$ coherent states and electro-dynamics is also established.

Conclusions and outlook

A link is established between electrodynamics, self-similarity and coherent states in QFT.

In space-time regions where the magnetic field may be approximated to be constant and the electric field is derivable from a harmonic potential, an isomorphism has been recognized to exist between electrodynamics and a set of damped and amplified oscillators, which are represented by $SU(1,1)$ squeezed (q-deformed) coherent states and are in turn isomorph to fractal self-similar structures.

Dissipation plays a central role and is at the origin of noncommutative geometry in the plane and is described by the deformed Hopf algebra.

Fractal-like structures with self-similarity properties appear as *macroscopic quantum systems* **generated by coherent** $SU(1,1)$ **quantum condensation processes at the microscopic level.**

Fractals emerge out of a process of *morphogenesis* **(***forms***) as the macroscopic manifestation of the dissipative, coherent quantum dynamics at the elementary level.**

An integrated vision of Nature based on the dynamics of coherence thus emerges.

It includes also the sector of high energy physics, with the coherent condensate structure of the vacuum and the recent discovery of the Higgs boson belongs to such a picture.

Nature appears to be shaped by coherence, rather than being organized in isolated compartments, in collections of isolated systems.

The dynamics of coherence appears to be the primordial origin of *codes*. **These are lifted from the (syntactic) level of pure information (à la Shannon) to the (semantic) level of** *meanings*, **expressions of coherent dynamical processes.**

Codes, (including the genetic DNA code) appear to be the *vehicles* **through which coherence propagate and manifest itself.**

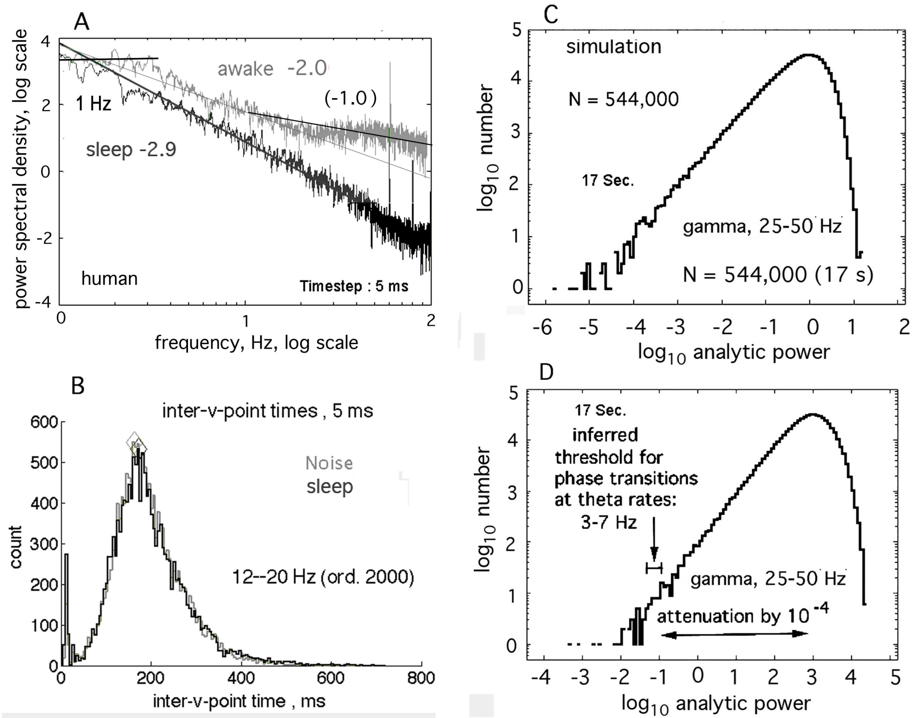
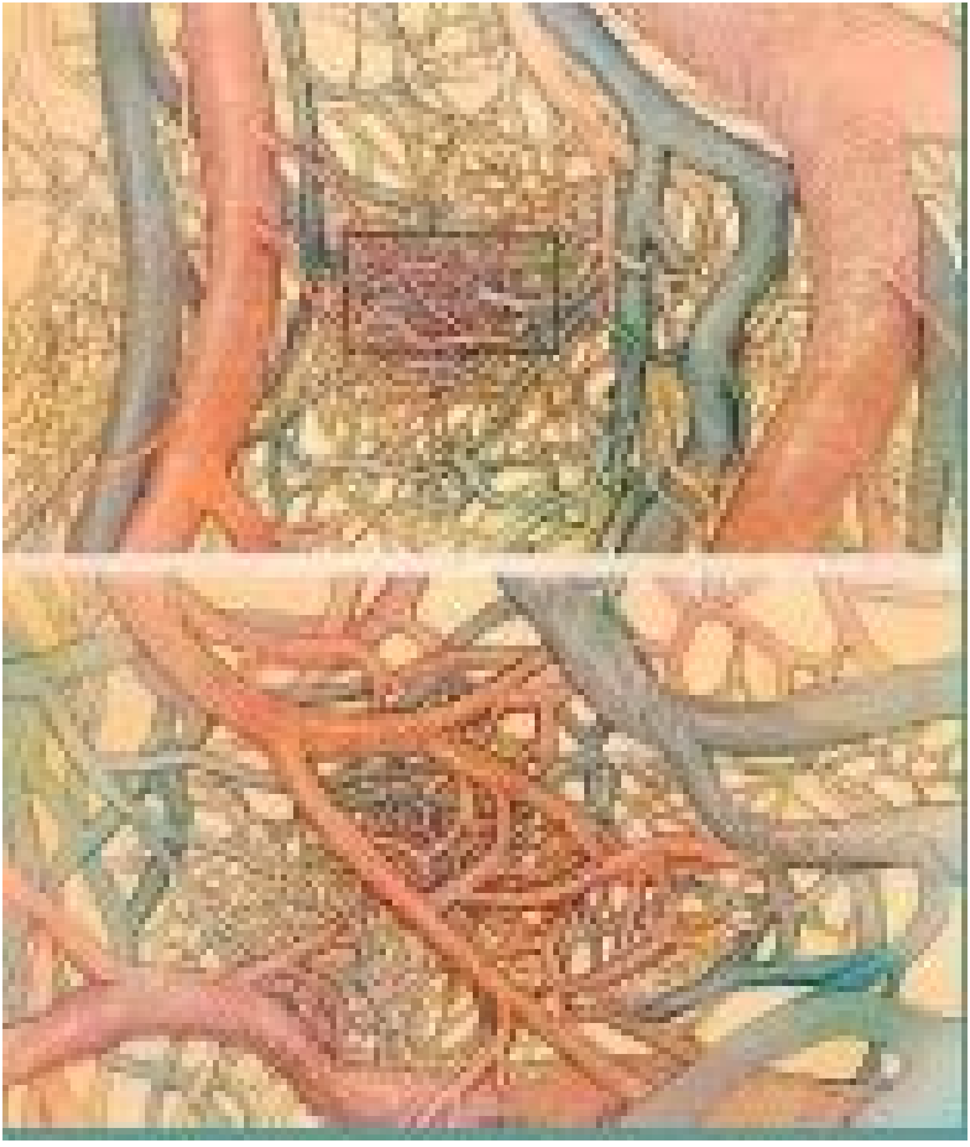
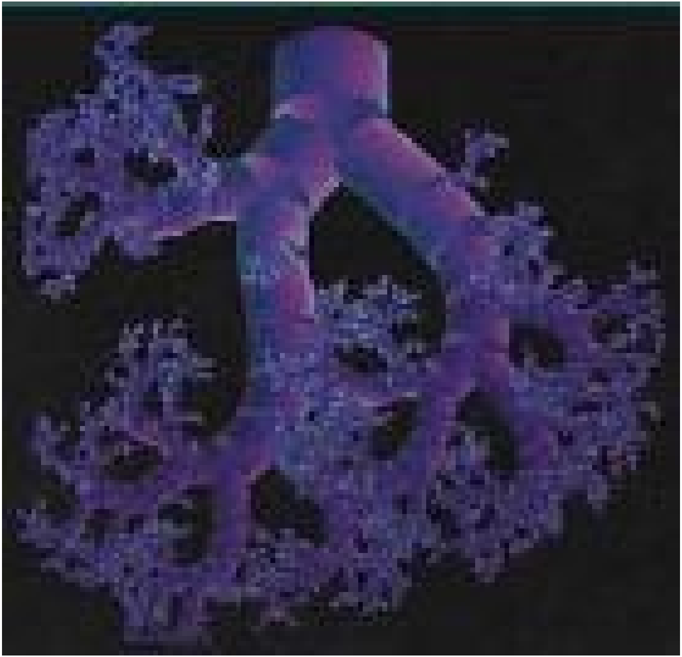


Figure 11. Evidence is summarized showing that the mesoscopic background activity conforms to scale-free, low-dimensional noise [Freeman et al., 2008]. Engagement of the brain in perception and other goal-directed behaviors is accompanied by departures from randomness upon the emergence of order (A), as shown by comparing PSD in sleep, which conforms to black noise, vs. PSD in an aroused state showing excess power in the theta ($3 - 7$ Hz) and gamma ($25 - 100$ Hz) ranges. B. The distributions of time intervals between null spikes of brown noise and sleep ECoG are superimposed. C,D. The distributions are compared of $\log_{10}$ analytic power from noise and ECoG. Hypothetically the threshold for triggering a phase transition is 10^{-4} down from modal analytic power. From [Freeman, O'Nuillain and Rodriguez, 2008 and Freeman and Zhai, 2009]





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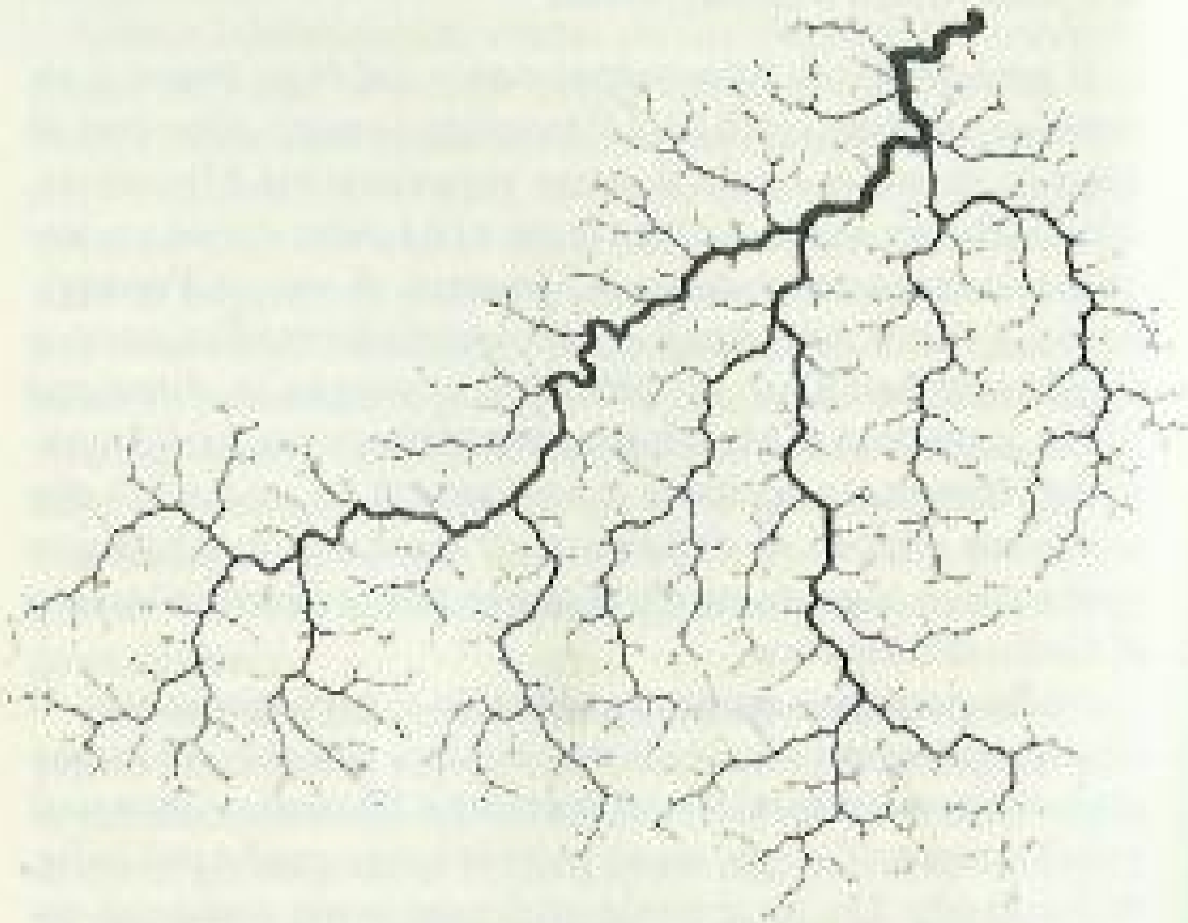
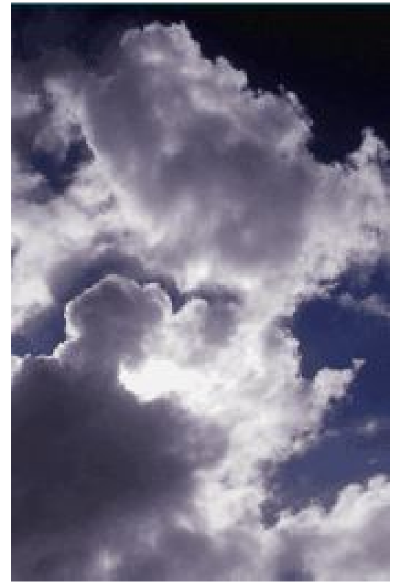


Figura 13. La struttura della rete fluviale del Fella, in Friuli. (L'immagine, riprodotta con il permesso di Ignacio Rodriguez-Iturbe e Andrea Rinaldo, è tratta da *Fractal River Basins*, Cambridge, Cambridge University Press, 1997.)





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Quantum Field Theory and its Macroscopic Manifestations

Boson Condensation, Ordered Patterns and Topological Defects

Massimo Blasone, Petr Jizba & Giuseppe Vitiello

Imperial College Press