Functional Renormalization Group Equations and Some Applications

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History of Field-Theoretic Renormalization

Renormalization as a technique to deal with infinities:

1938 Kramers *"The interaction between charged particles and the radiation field"*
1947 Bethe *"The electromagnetic shift of energy levels"*
1946-49 Tomonaga, Schwinger, Feynman, Dyson.

The renormalization group (RG):

1951 Stueckelberg & Petermann *"The normalization group in quantum theory"*
1955 Bogoliubov & Shirkov *"Charge renormalization group in quantum field theory"*

History of Field-Theoretic Renormalization

The RG as the physics of scales:

1954 Gell-Mann & Low *"Quantum electrodynamics at small distances"*
1970 Callan *"Broken scale invariance in scalar field theory"*
1970 Symanzik *"Small distance behavior in field theory and power counting"*

The RG as a bridge between (nonrenormalizable) theories:

1966 Kadanoff *"Scaling laws for Ising models near* T_C*"*
1971 Wilson *"Renormalization group and critical phenomena. 1. Renormalization group and the Kadanoff scaling picture"*
1972 Wegner *"Corrections to scaling laws"*

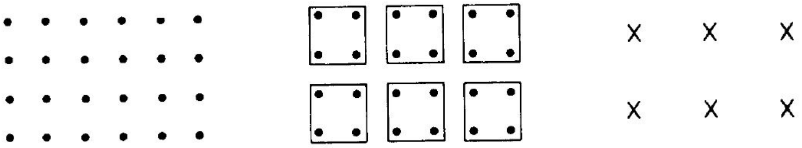
Kadanoff's Spin-Blocking

Statistical spin system (e.g. Ising model)

$$Z = \mathrm{Tr}_s e^{-\beta H(s)}$$

Take a block of k spins and assign a new spin to the block

$$T \, : \, (s_1, \ldots, s_k) \longrightarrow s'$$



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$$T \,:\, (s_1, \ldots, s_k) \longrightarrow s'$$

Define the dynamics of blocks

$$e^{-H'(s')} = \mathrm{Tr}_s \prod_{blocks} T\left(s'; s_i\right) e^{-\beta H(s)}$$

Spin-blocking must not change observables

$$\sum_{s'} T(s'; s_i) = 1 \longleftrightarrow \mathrm{Tr}_s e^{-\beta H(s)} = \mathrm{Tr}_{s'} e^{-\beta H'(s')}$$

Kadanoff's Spin-Blocking

The lattice spacing decreases of a factor k

The number of degrees of freedom decreases: $N' = Ne^{-kd}$
d being the dimension of the lattice

In the thermodynamic limit

$$\mathrm{Tr}_s e^{-\beta H(s)} = \left(\mathrm{Tr}_{s'} e^{-\beta H(s')}\right)^{\frac{N}{N'}}$$

and for Helmoltz free energy

$$F(H') = e^{kd} F(H)$$

Scaling at Second-Order Phase Transitions

Assume there is a fixed-point solution $H'_* = H_*$

Linearize and diagonalize the infinitesimal transformation δk around the fixed point

$$\left(H_* + \sum_i \mu_i O_i\right)' = H_* + \sum_i \mu_i \left(1 + \theta_i \delta k\right) O_i$$

For a finite k

$$\left(H_* + \sum_i \mu_i O_i\right)' = H_* + \sum_i \mu_i e^{\theta_i k} O_i$$

Widom's scaling hypothesis follows

$$F(\mu_i) = e^{-kd} F(\mu_i e^{\theta_i k})$$

Scaling at Second-Order Phase Transitions

The operators O_i are called

- relevant, $\theta_i > 0$
- irrelevant, $\theta_i < 0$
- marginal, $\theta_i = 0$

If $\mu_i = 0$ for all relevant operators, the system is at criticality and the $k \to \infty$ limit hits the fixed point

Normally the number of relevant operators is small!

At a second-order transition there two relevant operators

$$O_0 = 1, \qquad \theta_0 = d$$
$$O_T = ?, \qquad \theta_T = 1/\nu$$

Scaling at Second-Order Phase Transitions

Denoting

$$\begin{aligned}(1 - T/T_c) \ = \ \tau \quad &= \quad \mu_T \\ 2-\alpha \quad &= \quad d/\theta_T \\ e^{-kd} \quad &= \quad |\tau|^{2-\alpha} \\ \Delta_i \quad &= \quad \theta_i/\theta_T\end{aligned}$$

the free energy reads

$$F = \mu_0 + |\tau|^{2-\alpha} f^{\pm}_{\text{sing}}\left(\mu_i |\tau|^{-\Delta_i}\right)$$

μ_0 is the regular part of the free energy, while

$$f^{\pm}_{\text{sing}}(\dots) = F\left(\mu_0 = 0, \mu_T = \pm 1, \dots\right)$$

Wilson's Differential Formulation

Statistical field theory (Wick-rotated QFT) with a UV-cutoff Λ

$$Z = \int [\mathrm{d}\phi] \, e^{-S_\Lambda[\phi]}$$

Take a "block" of Fourier modes $\phi^>(p), \; p \in [\Lambda', \Lambda]$

$$\phi(p) \longrightarrow \phi^<(p) + \phi^>(p)$$
$$[0, \Lambda] \longrightarrow [0, \Lambda'] + [\Lambda', \Lambda]$$

Define the dynamics of $\phi^<$

$$e^{-S_{\Lambda'}[\phi^<]} = \int \left[\mathrm{d}\phi^>\right] e^{-S_\Lambda[\phi^< + \phi^>]}$$

Decimation does not change observables

$$\int [\mathrm{d}\phi] \, e^{-S_\Lambda[\phi]} = \int \left[\mathrm{d}\phi^<\right] e^{-S_{\Lambda'}[\phi^<]}$$

The RG is a Change of Variables

The continuous map $S_\Lambda \longrightarrow S_{\Lambda'}$ is an RG transformation.

Its generator is a change of parametrization of degrees of freedom.

Its fixed points, in the $\Lambda \to \infty/0$ limit, govern criticality, which **defines** scaling.

Scaling is universal. Given a certain scaling, one can group interactions into relevant and irrelevant.

Knowing the scaling, one can adapt the change of degrees of freedom to minimize the irrelevant interactions.

Effective Field Theories and the RG

Consider the example of QCD.

At high energy QCD is a free-field-theory of massless gluons and quarks.

At low energy QCD is the free-field-theory of massive hadrons and mesons.

The two RG fixed points have different scaling properties (different numbers of relevant parameters).

Effective descriptions in the two regimes have different degrees of freedom.

Observables are independent of the description we choose:
the two theories must match at intermediate energy scales!

The map between effective theories is the RG.

History of the Exact RG

1970 Wilson presents the first exact RG equation at the Irvine Conference

Wilson Exact RG Equation

The Renormalization group and the epsilon expansion

K.G. Wilson (Princeton, Inst. Advanced Study & Cornell U., LNS) , John B. Kogut (Princeton, Inst. Advanced Study)

Jul 1973 - 126 pages

Phys.Rept. 12 (1974) 75-200
DOI: 10.1016/0370-1573(74)90023-4

Abstract (Elsevier)
The modern formulation of the renormalization group is explained for both critical phenomena in classical statistical mechanics and quantum field theory. The expansion in $\epsilon = 4 - d$ is explained [d is the dimension of space (statistical mechanics) or space-time (quantum field theory)]. The emphasis is on principles, not particular applications. Sections 1–8 provide a self-contained introduction at a fairly elementary level to the statistical mechanical theory. No background is required except for some prior experience with diagrams. In particular, a diagrammatic approximation to an exact renormalization group equation is presented in sections 4 and 5; sections 6–8 include the approximate renormalization group recursion formula and the Feyman graph method for calculating exponents. Sections 10–13 go deeper into renormalization group theory (section 9 presents a calculation of anomalous dimensions). The equivalence of quantum field theory and classical statistical mechanics near the critical point is established in section 10; sections 11–13 concern problems common to both subjects. Specific field theoretic references assume some background in quantum field theory. An exact renormalization group equation is presented in section 11; sections 12 and 13 concern fundamental topological questions.

Keyword(s): INSPIRE: lectures | mechanics: statistics | renormalization | group theory | transformation | feynman graph | bibliography

Wilson Exact RG Equation

"The formal discussion of consequences of the renormalization group works best if one has a differential form of the renormalization group transformation."

$$\frac{\partial \mathcal{H}_t}{\partial t} = \int_q \frac{\partial \alpha_q(t)}{\partial t} \left\{ \frac{\delta \mathcal{H}_t}{\delta \sigma''_q} \frac{\delta \mathcal{H}_t}{\delta \sigma''_{-q}} + \frac{\delta^2 \mathcal{H}_t}{\delta \sigma''_q \, \delta \sigma''_{-q}} + \sigma''_q \frac{\delta \mathcal{H}_t}{\delta \sigma''_q} + \text{const.} \right\}$$

"A longer range possibility is that one will be able to develop approximate forms of the transformation which can be integrated numerically; if so, one might be able to solve problems which cannot be solved any other way."

"These equations are very complicated so they will not be discussed in great detail."

Wegner-Houghton Exact RG Equation

Renormalization group equation for critical phenomena

Franz J. Wegner (Julich, Forschungszentrum) , Anthony Houghton (Brown U.)

Oct 1972 - 12 pages

Phys.Rev. A8 (1973) 401-412
DOI: 10.1103/PhysRevA.8.401

Abstract (APS)
An exact renormalization equation is derived by making an infinitesimal change in the cutoff in momentum space. From this equation the expansion for critical exponents around dimensionality 4 and the limit $n=\infty$ of the n-vector model are calculated. We obtain agreement with the results of Wilson and Fisher, and with the spherical model.

Perturbative Solution of the Exact RG

"Here a renormalization-group equation is derived by eliminating the Fourier components of the order parameter in an infinitesimally small shell in k space."

"To demonstrate the usefulness of our equation, we consider
(a) the expansion around dimensionality 4 for the n-vector model and rederive critical exponents to order ϵ and η to order ϵ^2,
(b) the limit $n = \infty$ of the n-vector model."

History of the Functional RG

Radiative Corrections as the Origin of Spontaneous Symmetry Breaking

Sidney R. Coleman, Erick J. Weinberg (Harvard U.)

1973 - 23 pages

Phys.Rev. D7 (1973) 1888-1910
Also in *Mohapatra, R. N. (Ed.), Lai, C. H. (Ed.): Gauge Theories Of Fundamental Interactions*, 57-79
In *Abbott, L.F. (ed.), Pi, S.Y. (ed.): Inflationary cosmology*, 589-611.
DOI: 10.1103/PhysRevD.7.1888

Abstract (APS)
We investigate the possibility that radiative corrections may produce spontaneous symmetry breakdown in theories for which the semiclassical (tree) approximation does not indicate such breakdown. The simplest model in which this phenomenon occurs is the electrodynamics of massless scalar mesons. We find (for small coupling constants) that this theory more closely resembles the theory with an imaginary mass (the Abelian Higgs model) than one with a positive mass; spontaneous symmetry breaking occurs, and the theory becomes a theory of a massive vector meson and a massive scalar meson. The scalar-to-vector mass ratio is computable as a power series in e, the electromagnetic coupling constant. We find, to lowest order, m2(S)m2(V)=(32π)(e24π). We extend our analysis to non-Abelian gauge theories, and find qualitatively similar results. Our methods are also applicable to theories in which the tree approximation indicates the occurrence of spontaneous symmetry breakdown, but does not give complete information about its character. (This typically occurs when the scalar-meson part of the Lagrangian admits a greater symmetry group than the total Lagrangian.) We indicate how to use our methods in these cases.

One Loop Effective Potential

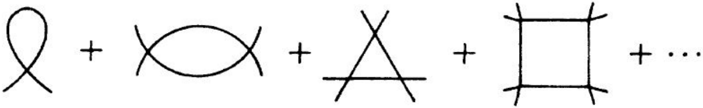

FIG. 2. The one-loop approximation for the effective potential.

$$V = \frac{\lambda}{4!}\varphi_c{}^4 - \tfrac{1}{2}B\varphi_c{}^2 - \frac{1}{4!}C\varphi_c{}^4 + \tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ln\left(1+\frac{\lambda\varphi_c{}^2}{2k^2}\right),$$

One Loop Effective Potential

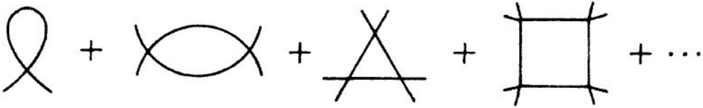

FIG. 2. The one-loop approximation for the effective potential.

$$V = \frac{\lambda}{4!}\varphi_c{}^4 + \tfrac{1}{2}B\varphi_c{}^2 + \frac{1}{4!}C\varphi_c{}^4 + \frac{\lambda\Lambda^2}{64\pi^2}\varphi_c{}^2 + \frac{\lambda^2\varphi_c{}^4}{256\pi^2}\left(\ln\frac{\lambda\varphi_c{}^2}{2\Lambda^2} - \frac{1}{2}\right)$$

Wilson's Speculation

"A longer range possibility is that one will be able to develop approximate forms of the transformation which can be integrated numerically; if so, one might be able to solve problems which cannot be solved any other way."

Local Potential Approximation

References (11) Citations (80) Files Plots

Approximate Renormalization Group Based on the Wegner-Houghton Differential Generator

J.F. Nicoll, T.S. Chang, H.E. Stanley (MIT)

Aug 26, 1974 - 3 pages

Phys.Rev.Lett. 33 (1974) 540-543
DOI: 10.1103/PhysRevLett.33.540

Abstract (APS)
We give an approximate renormalization-group formulation which parallels that of Wilson. The group generator represents the momentum-independent limit of the differential generator of Wegner and Houghton. The eigenfunctions near the Gaussian point are computed for all spin dimensions n and lattice dimensions d, including d=2. The nontrivial fixed-point Hamiltonian in dimensions near d=2O(O−1), together with the eigenvalues near that nontrivial fixed point, are found explicitly to first order in εO≡O(2−d)+d for all values of n and the order O. Odd-dominated Ising systems and corresponding expansions in εO−12 are also treated.

Local Potential Approximation (LPA)

Beginning of functional truncations of exact RG equations

Local Potential Approximation

Upon projection on vanishing momenta the ERGE becomes a PDE

$$\dot{H} = dH + (2-d)x\frac{\partial H}{\partial x} + \frac{d}{2}\left[1 - \frac{1}{n}\ln\left(1 + \frac{\partial H}{\partial x}\right) + \frac{1}{n}\ln\left(1 + \frac{\partial H}{\partial x} + 2x\frac{\partial^2 H}{\partial x^2}\right)\right]$$

"To solve (1) [this equation], the Hamiltonian can be expanded in terms of any complete set of functions; the expansion functions should be chosen to simplify the problem under consideration. "

"[...] the eigenfunctions of (1) when (1) is linearized about the Gaussian fixed point, $H = 0$", (here Laguerre polynomials)

Scaling-Fields Expansion (Wegner 1972)

"If H is expanded in powers of x, the resulting equations, while not appropriate for general ϵ-analysis, are essentially triangular"

History of the Exact RG

So far either ϵ-expansion or $n \to \infty$

Beyond perturbation theory?

Numerical Methods Coming

Calculation of the Critical Exponent eta via Renormalization-Group Recursion Formulas

Geoffrey R. Golner (Cornell U., LNS)

Jul 1, 1973 - 6 pages

Phys.Rev. B8 (1973) 339-345
DOI: 10.1103/PhysRevB.8.339

Abstract (APS)
This paper presents an extension of Wilson's renormalization-group calculation of Ising-model critical exponents to include calculation of the critical exponent η. New recursion formulas are derived using the simplest set of consistent approximations which allow a nonzero η. They are intended to demonstrate, qualitatively, how nonzero values for η are consistent with the renormalization-group approach; they do not represent systematic, quantitative improvements to Wilson's earlier calculation of the exponents ν and γ. The equations are solved both by ε expansion about four dimensions and by numerical integration in three dimensions. To order ε2 we obtain η=0.05ε2. Numerical results in three dimension are η=0.058, ν=0.588, and γ=1.14. The relation γ=(2−η)ν is confirmed.

Functional, exact, but not continuous (discrete RG steps)

The Beginnings of the Derivative Expansion

A consistent treatment of the new terms requires a larger space of Hamiltonians of the form

$$\mathcal{H} = -\int_{\vec{x}} P(s(\vec{x})) - \tfrac{1}{2}K \int_{\vec{x}} R(s(\vec{x}))[\vec{\nabla} s(\vec{x})]^2 ,$$

where $R(s(\vec{x}))$ is an arbitrary function of $s(\vec{x})$.

First Nonperturbative Applications

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Renormalization-Group Calculation of Critical Exponents in Three Dimensions

Geoffrey R. Golner, Eberhard K. Riedel (Duke U.)

Apr 7, 1975 - 3 pages

Phys.Rev.Lett. 34 (1975) 856-859
DOI: 10.1103/PhysRevLett.34.856

Abstract (APS)
A scaling-field representation of Wilson's exact renormalization-group equation is derived and used for the approximate calculation of critical exponents for the continuous-spin Ising model in three dimensions. The truncation of the hierarchy of scaling-field equations that retains only the four most relevant scaling fields yields the critical exponents $\nu \approx 0.60$, $\eta \approx 0.06$, and $\Delta 1 \approx -0.47$ (where $\Delta 1$ denotes Wortis's correction-to-scaling exponent).

First Example of Truncation-Induced Problems

"The parameter A denotes the arbitrary normalization of the kinetic energy term in the Hamiltonian"

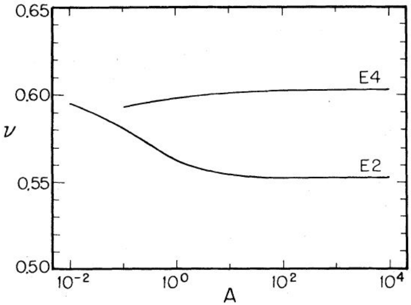


FIG. 1. Critical exponent ν as a function of A as obtained from the sets of two scaling-field equations ($E2$) and four scaling-field equations ($E4$).

The Beginnings of the 1PI Exact RG

An Exact One Particle Irreducible Renormalization Group Generator for Critical Phenomena

J.F. Nicoll, T.S. Chang (MIT, LNS)

1977 - 3 pages

Phys.Lett. A62 (1977) 287-289
DOI: 10.1016/0375-9601(77)90417-0

Abstract (Elsevier)
An exact one-particle-irreducible renormalization-group generator for critical phenomena is derived by an infinitesimal saddle-point expansion. This replaces the usual field-theoretic loop-expansion for the free energy and Green's functions with an explicit differential equation.

$$\frac{\partial A}{\partial l} = \frac{1}{2}\,\mathrm{tr}\ln\left[A_{ss'} - A_{sg}(A^{-1})_{gg'}A_{g's}\right]$$

Towards Precision

Critical exponents by the scaling-field method: The isotropic N-vector model in three dimensions

Kathie E. Newman (Notre Dame U.) , Eberhard K. Riedel (Washington U., Seattle)

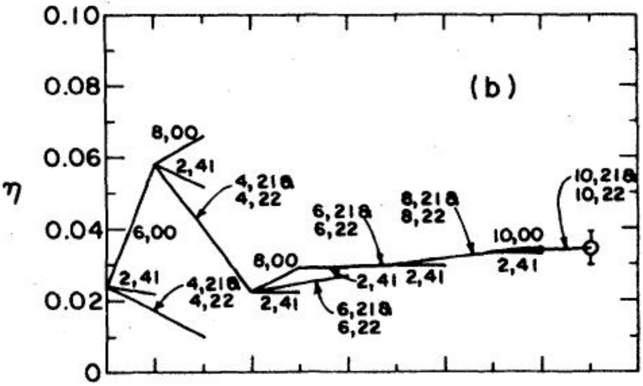
Dec 1, 1984 - 23 pages

Phys.Rev. B30 (1984) 6615-6638
DOI: 10.1103/PhysRevB.30.6615

Abstract (APS)
A numerical technique, termed the scaling-field method, is developed for solving by successive approximation Wilson's exact renormalization-group equation for critical phenomena in three-dimensional spin systems. The approach uses the scaling-field representation of the Wilson equation derived by Riedel, Golner, and Newman. A procedure is proposed for generating in a nonperturbative and unbiased fashion sequences of successively larger truncations to the infinite hierarchy of scaling-field equations. A "principle of balance" is introduced and used to provide a self-consistency criterion. The approach is then applied to the isotropic N-vector model. Truncations to order 13 (10, when N=1) scaling-field equations yield the leading critical exponents, ν and η, and several of the correction-to-scaling exponents, Δm, to high precision. Results for N=0, 1, 2, and 3 are tabulated. For the Ising case (N=1), the estimates ν=0.626±0.009, η=0.040±0.007, and Δ1≡Δ400=0.54±0.05 are in good agreement with recent high-temperature-series results, though exhibiting larger confidence limits at the present level of approximation. For the first time, estimates are obtained for the second and third correction-to-scaling exponents. For example, for the Ising model the second "even" and first "odd" correction-to-scaling exponents are Δ422=1.67±0.11 and Δ500=1.5±0.3, respectively. Extensions necessary to improve the accuracy of the calculation are discussed, while applications of the approach to anisotropic N-vector models are described elsewhere. Finally, the scaling-field method is compared with other techniques for the high-precision calculation of critical phenomena in three dimensions, i.e., high-temperature-series, Monte Carlo renormalization-group, and field-theoretic perturbation expansions.

Towards Precision

Convergence studies for truncation strategies



The derivative expansion: systematics and $O(\partial^4)$

ations (47) Files Plots

Nonperturbative Renormalization Group Calculations for Continuum Spin Systems

Geoffrey R. Golner (Maharishi U. of Management)

Jun 1985 - 16 pages

Phys.Rev. B33 (1986) 7863-7866
DOI: 10.1103/PhysRevB.33.7863
MIU-THP-85/011

Abstract (APS)
Wilson's exact smooth-cutoff renormalization-group (RG) equation for continuum spin Landau-Ginsburg models is shown to be equivalent to an easily constructed, infinite set of partial differential equations that provide a natural system of successive approximation for numerical calculation. It differs from other RG approaches in that an infinite number of couplings are included at each level of approximation. By way of illustration, preliminary results are presented for Ising-model critical exponents in three dimensions.

Perturbative Renormalizability

Renormalization and Effective Lagrangians

Joseph Polchinski (Harvard U.)

Apr 1983 - 27 pages

Nucl.Phys. B231 (1984) 269-295
(1984)
DOI: 10.1016/0550-3213(84)90287-6
HUTP-83-A018

Abstract (Elsevier)
There is a strong intuitive understanding of renormalization, due to Wilson, in terms of the scaling of effective lagrangians. We show that this can be made the basis for a proof of perturbative renormalization. We first study renormalizability in the language of renormalization group flows for a toy renormalization group equation. We then derive an exact renormalization group equation for a four-dimensional λø 4 theory with a momentum cutoff. We organize the cutoff dependence of the effective lagrangian into relevant and irrelevant parts, and derive a linear equation for the irrelevant part. A lengthy but straightforward argument establishes that the piece identified as irrelevant actually is so in perturbation theory. This implies renormalizability. The method extends immediately to any system in which a momentum-space cutoff can be used, but the principle is more general and should apply for any physical cutoff. Neither Weinberg's theorem nor arguments based on the topology of graphs are needed.

Perturbative Renormalizability

1988 Wieczerkowski *"Symanzik's Improved Actions From the Viewpoint of the Renormalization Group"*

1990 Keller, Kopper, Salmhofer *"Small distance behavior in field theory and power counting"*

1991 Keller, Kopper *"Perturbative renormalization of QED via flow equations"*

"for suitable renormalization conditions [...] the violation of the Ward identity goes to zero as $\Lambda_0 \to \infty$"

1992 Bonini, D'Attanasio, Marchesini *"Perturbative renormalization and infrared finiteness in the Wilson renormalization group: the massless scalar case"* Exact RG equation for the 1PI effective action!

1PI Exact RG Equation

The Exact renormalization group and approximate solutions

Tim R. Morris (CERN)

Aug 1993 - 44 pages

Int.J.Mod.Phys. A9 (1994) 2411-2450
DOI: 10.1142/S0217751X94000972
CERN-TH-6977-93, SHEP-92-93-27
e-Print: hep-ph/9308265 | PDF

Abstract
We investigate the structure of Polchinski's formulation of the flow equations for the continuum Wilson effective action. Reinterpretations in terms of I.R. cutoff greens functions are given. A promising non-perturbative approximation scheme is derived by carefully taking the sharp cutoff limit and expanding in `irrelevancy' of operators. We illustrate with two simple models of four dimensional $\lambda\varphi$ 4 theory: the cactus approximation, and model incorporating the first irrelevant correction to the renormalized coupling. The qualitative and quantitative behaviour give confidence in a fuller use of this method for obtaining accurate results.

FLow equations for N point functions and bound states

Ulrich Ellwanger (Heidelberg U.)

Sep 24, 1993 - 16 pages

Z.Phys. C62 (1994) 503-510
Frankenhausen 1993, Proceedings, Quantum field theoretical aspects of high energy physics 206-211, and Heidelberg U. - HD-THEP-93-30 (93,rec.Aug.) 19 p
DOI: 10.1007/BF01555911
Conference: C93-09-20.1, p.206-211
Proceedings
HD-THEP-93-30
e-Print: hep-ph/9308260 | PDF

Abstract
We discuss the exact renormalization group or flow equation for the effective action and its decomposition into one particle irreducible N point functions. With the help of a truncated flow equation for the four point function we study the bound state problem for scalar fields. A combination of analytic and numerical methods is proposed, which is applied to the Wick-Cutkosky model and a QCD-motivated interaction. We present results for the bound state masses and the Bethe-Salpeter wave function. (Figs. 1-4 attached as separate uuencoded post-script files.)

The Vertex Expansion

$$\varphi(q) \to \varphi_i,\ \Gamma_N(q_1 \ldots q_N) \to \Gamma_N^{i_1 \ldots i_N}, \int \frac{d^4q}{(2\pi)^{''}} \to \sum_i \text{, etc.}$$

$$\begin{aligned}\Gamma_k^{\Lambda}(\varphi) &= \frac{1}{2}\varphi_i\varphi_j\Gamma_2^{ij} + \frac{1}{4!}\varphi_i\varphi_j\varphi_k\varphi_l\Gamma_4^{ijkl} \\ &\quad + \frac{1}{6!}\varphi_i\varphi_j\varphi_k\varphi_l\varphi_m\varphi_n\Gamma_6^{ijklmn} + \ldots\end{aligned}$$

The Wetterich Story: Continuous Spin Blocking

Average Action and the Renormalization Group Equations

C. Wetterich (DESY)

Dec 1989 - 56 pages

Nucl.Phys. B352 (1991) 529-584
(1991)
DOI: 10.1016/0550-3213(91)90099-J
DESY-89-168

Abstract (Elsevier)
We formulate an effective action Γ k for averages of fields taken within a volume of size k − d . In contrast to the block-spin approach on the lattice we work in continuous (euclidean) space, preserving all symmetries. We establish how expectation values of operators with momenta smaller than k can be computed from Γ k . The average action at different scales is related by an exact renormalization group equation. We apply these ideas to the N-component ϕ 4 theory in the spontaneously broken phase and derive the one-loop renormalization group equations for the average potential. The average potential becomes convex as k → 0.

The Wetterich Story: One-Loop RG Improvement

Apr 1990 Wetterich *" Quadratic Renormalization of the Average Potential and the Naturalness of Quadratic Mass Relations for the Top Quark"*

Aug 1991 Wetterich *"The Average action for scalar fields near phase transitions"*

Feb 1992 Wetterich, Bornholdt, *"Selforganizing criticality, large anomalous mass dimension and the gauge hierarchy problem"*

Mar 1992 Wetterich, Reuter, *"Average action for the Higgs model with Abelian gauge symmetry"*

Apr 1992 Wetterich, Tetradis, *"Scale dependence of the average potential around the maximum in ϕ^4 theories"*

Jul 1992 Wetterich, Tetradis, *"The high temperature phase transition for ϕ^4 theories"*

Dec 1992 Wetterich, Bornholdt, *"Average action for models with fermions"*

One-Loop RG Improvement is Exact!

Exact evolution equation for the effective potential

Christof Wetterich (Heidelberg U.)

Dec 29, 1992 - 9 pages

Phys.Lett. B301 (1993) 90-94
DOI: 10.1016/0370-2693(93)90726-X
HD-THEP-92-61

Abstract (Elsevier)
We derive a new exact evolution equation for the scale dependence of an effective action. The corresponding equation for the effective potential permits a useful truncation. This allows one to deal with the infrared problems of theories with massless modes in less than four dimensions which are relevant for the high temperature phase transition in particle physics or the computation of critical exponents in statistical mechanics.

Same as Bonini et al., Morris, Ellwanger.

Almost the same as Nicoll&Chang

The Lesson To Be Learned

Exact + As Simple As Possible = One Loop!

BUT: the action must be the most general one

Take a one loop formula with a mass-like regulator k^2

$$\Gamma_m[\phi] = S[\phi] + \frac{1}{2}\mathrm{Tr}\log\left(S^{(2)}[\phi] + k^2\right)$$

As in the Callan-Symanzik RG, consider k as the RG scale
and compute the beta-functional by $\partial_t = k\partial_k$

$$\partial_t\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\left(S^{(2)}[\phi] + k^2\right)^{-1}\partial_t k^2\right]$$

and identify as usual bare couplings (S) with running ones (Γ)

The same holds for a momentum-dependent mass $k \to R_k(p^2)$

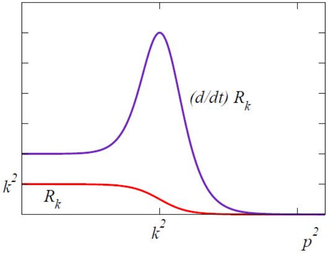
The Lesson To Be Learned

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr}\left[\left(\Gamma^{(2)}[\phi] + R_k\right)^{-1} \partial_t R_k\right]$$

Proof of Exactness

$$e^{W_k[J]} = \int [\mathrm{d}\varphi] e^{-S[\varphi] - \frac{1}{2}\varphi \cdot R_k \cdot \varphi + J \cdot \varphi}$$

Wilsonian integration shell-by-shell



(essentially) Polchinski equation

$$-\partial_t W_k[J] = \frac{1}{2}\mathrm{Tr}\left[\frac{\delta^2 W_k}{\delta J \delta J}\partial_t R_k\right] + \frac{1}{2}\mathrm{Tr}\left[\frac{\delta W_k}{\delta J}\partial_t R_k \frac{\delta W_k}{\delta J}\right]$$

Proof of Exactness

$$e^{W_k[J]} = \int [\mathrm{d}\varphi] e^{-S[\varphi] - \frac{1}{2}\varphi \cdot R_k \cdot \varphi + J \cdot \varphi}$$

(essentially) Polchinski equation

$$-\partial_t W_k[J] = \frac{1}{2} \mathrm{Tr}\left[\frac{\delta^2 W_k}{\delta J \delta J} \partial_t R_k \right] + \frac{1}{2} \mathrm{Tr}\left[\frac{\delta W_k}{\delta J} \partial_t R_k \frac{\delta W_k}{\delta J} \right]$$

Legendre transform

$$\Gamma_k[\phi] = \operatorname*{ext}_J \left(J \cdot \phi - W_k[J] \right) - \frac{1}{2} \phi \cdot R_k \cdot \phi$$

flow of the average effective action

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr}\left[\left(\Gamma^{(2)}[\phi] + R_k \right)^{-1} \partial_t R_k \right]$$

User's Manual

1. Decide what you want to compute
2. Choose the truncation strategy accordingly
   - Scaling Field Expansion
   - Derivative Expansion
   - Vertex Expansion
   - ...
3. Study the dependence of universal predictions on R_k and optimize
4. Estimate truncation errors

Personal Selection of Applications: Particle Theory

1993 Reuter, Wetterich, *"Running gauge coupling in three-dimensions and the electroweak phase transition "*

1994 Halpern, Huang, *"Fixed point structure of scalar fields"*

2001 Codello, Percacci, *"Fixed Points of Nonlinear Sigma Models in* $d > 2$*"*

2002 Gies, *"Running coupling in Yang-Mills theory: A flow equation study"*

2002 Gies, Jaeckel, Wetterich, *"Towards a renormalizable standard model without fundamental Higgs scalar"*

2010 Gies, Scherer, *"Asymptotic safety of simple Yukawa systems"*

2013 Gies, Gneiting, Sondenheimer, *"Higgs Mass Bounds from Renormalization Flow for a simple Yukawa model "*

Personal Selection of Applications: Quantum Gravity

1996 Reuter, *"Nonperturbative evolution equation for quantum gravity"*

2000 Bonanno, Reuter, *"Renormalization group improved black hole space-times"*

2003 Percacci, Perini, *"Asymptotic safety of gravity coupled to matter "*

2008 Codello, Percacci, Rahmede, *"Ultraviolet properties of $f(R)$-gravity"*

2009 Weinberg, *"Asymptotically safe inflation"*

2009 Shaposhnikov, Wetterich, *"Asymptotic safety of gravity and the Higgs boson mass"*

Personal Selection of Applications: IR QCD

1996 Jungnickel, Wetterich, *"Effective action for the chiral quark-meson model"*

2001 Gies, Wetterich, *"Renormalization flow of bound states"*

2004 Pawlowski, Litim, von Smekal, *"Infrared behavior and fixed points in Landau gauge QCD"*

2010 Braun, Gies, Pawlowski *"Quark Confinement from Color Confinement"*

2014 Tripolt, von Smekal, Wambach, *"Flow equations for spectral functions at finite external momenta"*

2015 Mitter, Pawlowski, Strodthoff, *"Chiral symmetry breaking in continuum QCD"*

And Much More

- All kinds of critical phenomena: disorder, long range, membranes
- Out-of-equilibrium: dissipation, transport, avalanches, turbolence
- Few/many body, closed/open systems: Efimov, cold atoms, nuclei
- Strongly interacting electrons and condensed matter: graphene, superconductors, topological transitions
- Supersymmetric field theories
- Group field theories and matrix models
- Holography
- de Sitter (in)stability and QFT in curved spaces